

Integrable Submodels of Nonlinear σ -models and Their Generalization

Tatsuo Suzuki

A dissertation submitted for the degree of
Doctor of Science at Waseda University
March, 2001

Preface

The nonlinear σ -model is a field theory whose action is a energy functional of maps from a space-time to a Riemannian manifold. This theory is used in particle physics and condensed-matter physics as low-energy effective theories. The solutions of its equation of motion are called harmonic maps, which are important object in geometry. If the dimension of the space-time is $1+n$ (this means one time-variable and n space-variables) and the Riemann manifold is a Grassmann manifold, it is called the nonlinear Grassmann model in $(1+n)$ dimensions.

The nonlinear Grassmann model in $(1+1)$ dimensions is a very interesting theory and is investigated in huge amount of papers. In particular, it has integrable structures such as a Lax-pair, an infinite number of conserved currents, a wide class of exact solutions and so on [Zak]. Moreover, this model carries instantons which exhibit many similarities to instantons of the four-dimensional Yang-Mills theory [Zak], [Raj]. In view of these rich structures, it may be natural to investigate the structures of the nonlinear Grassmann models in higher dimensions.

We are interested in integrable structures of the nonlinear Grassmann models in higher dimensions. However, we find that it is hard to study the higher-dimensional models in a similar way as $(1+1)$ -dimensional one because it is difficult to extend the concepts of integrability such as zero-curvature

conditions to higher dimensions.

In these circumstances, O. Alvarez, L. A. Ferreira and J. S. Guillen proposed a new approach to higher-dimensional integrable theories [AFG1] (see also [AFG2]) in 1997. They defined a local integrability condition in higher dimensions as the vanishing of a curvature of a trivial bundle over a path-space. They investigated the condition for the nonlinear \mathbf{CP}^1 -model in $(1+2)$ dimensions. Then they found an integrable submodel of the nonlinear \mathbf{CP}^1 -model in $(1+2)$ dimensions. We call it the \mathbf{CP}^1 -submodel for short. The \mathbf{CP}^1 -submodel possesses an infinite number of conserved currents and a wide class of exact solutions. Moreover, solutions of the \mathbf{CP}^1 -submodel are also those of the nonlinear \mathbf{CP}^1 -model. Thus, by analysing the submodel, we can investigate hidden structures of the original model.

After AFG's proposal, the theory of integrable submodels has been generalized and applied to some interesting models in physics to find new non-perturbative structures [FS1], [FS2], [GMG], [FHS1], [FHS2], [FL], [AFZ1], [AFZ2], [Suz], [FG], [B], [FR]. Therefore it is expected that the theory of submodels open a new way to develop exact methods in higher-dimensional field theories.

In this thesis, we investigate various integral submodels and generalize them. We use the word “integrable” in the sense of possessing an infinite number of conserved currents.

This thesis consists of three parts. In part I, we study the submodel of the nonlinear \mathbf{CP}^1 -model and the related submodels in $(1+2)$ dimensions. We give an explicit formula of the conserved currents with a discrete parameter by using a more general potential than one introduced in [AFG1]. This generalized formula implies the conserved currents of the \mathbf{CP}^1 -submodel which are associated with the spin j representation of $SU(2)$. Next, we define submod-

els of related models to the nonlinear \mathbf{CP}^1 -model. The generalized formula mentioned above also implies the conserved currents of these submodels.

In part II, we construct integrable submodels of the nonlinear Grassmann models in any dimension. We call them the Grassmann submodels. We define a Grassmann manifold as a set of projection matrices. Then we find that the tensor product of matrices is a key to define integrable submodels. To show that our submodels are integrable, we construct an infinite number of conserved currents in two ways. One is that we make full use of the Noether currents of the nonlinear Grassmann models. The other is that we use a method of multiplier. We pull back a 1-form on the Grassmann manifold to the Minkowski space. Then we study conditions which make the pull-backed form a conserved current. These conditions are especially noteworthy in the case of the \mathbf{CP}^1 -submodel. As a result, we find a generic form of the conserved currents of the \mathbf{CP}^1 -submodel. Next we investigate symmetries of the Grassmann submodel. By using the symmetries, we can construct a wide class of exact solutions for our submodels.

In part III, keeping some properties of our submodels, we generalize our submodels to higher-order equations. The generic form of the conserved currents and symmetries of our submodels studied in part II become clearer by this generalization. First we prepare the Bell polynomials and the generalized Bell polynomials which play the most important roles in our theory of generalized submodels. The Bell polynomials were introduced by E. T. Bell [Bell]. These polynomials are used in the differential calculations of composite functions. Next we generalize the \mathbf{CP}^1 -submodel to higher-order equations. By introducing a “symbol” of the Bell polynomials, we can construct an infinite number of conserved currents of the generalized submodels by some simple calculations only. To construct exact solutions of our gener-

alized submodels, we generalize a method discovered by V. I. Smirnov and S. L. Sobolev. Moreover, we also investigate symmetries of the generalized submodels. Lastly we generalize the Grassmann submodel to higher-order equations. By using the generalized Bell polynomials, we can show that the generalized Grassmann submodels are also integrable. As a result, we obtain a hierarchy of systems of integrable equations in any dimension which includes Grassmann submodels. These results lead to the conclusion that the integrable structures of our generalized submodels are closely related to some fundamental properties of the Bell polynomials.

Acknowledgements

I would like to express my deep gratitude to my advisor Professor Tosiaki Kori for his encouragement during the time of the thesis. I wish to express my hearty gratitude to my co-author Professor Kazuyuki Fujii for many discussions and valuable advices, and to my co-author Mr. Yasushi Homma for his contribution to solve some important problems. Finally, I would like to thank Professor Kimio Ueno and Professor Masayoshi Tsutsumi for their critical reading of my manuscript as the members of the doctoral committee.

Tatsuo Suzuki

Contents

Preface	i
I A Submodel of the Nonlinear CP^1-Model and Related Submodels in $(1 + 2)$ Dimensions	viii
1 A Submodel of the Nonlinear CP^1-Model	1
1.1 AFG's Local Integrability Condition in $(1 + 2)$ Dimensions . .	1
1.2 AFG's Definition of the CP^1 -submodel	3
1.3 Formulas for a Generalized Potential	7
1.4 Conserved Currents associated with Spin j Representation of $SU(2)$	11
2 Submodels of Related Models	13
2.1 CP^1 -like Models and their Submodels	13
2.2 QP^1 -model and its submodel	15

II	A Submodel of the Nonlinear Grassmann Model in Any Dimension	20
3	A Definition of a Submodel of the Nonlinear Grassmann Model	21
3.1	Grassmann Manifolds (Projection-Matrix Expression)	21
3.2	The Nonlinear Grassmann Sigma Model	24
3.3	A Definition of a Submodel	28
4	Conserved Currents of the Grassmann Submodel	30
4.1	Tensor Noether Currents	30
4.2	Pull-Backed 1-Form	34
5	Symmetries of the Grassmann Submodel	37
III	A Generalization of the Grassmann Submodel to Higher-Order Equations	39
6	The Bell Polynomials	40
6.1	The Bell Polynomials of One Variable	40
6.2	The Generalized Bell Polynomials	44
7	Higher-Order CP^1-submodels	47
7.1	Definitions of Higher-Order CP^1 -submodels	47
7.2	Conserved Currents	49
7.3	Symmetries of Higher-Order CP^1 -submodels	55
7.3.1	A Generalization of Smirnov and Sobolev's Construction	55
7.3.2	Symmetries and a Property of the Bell Matrix	60

8	A Higher-Order Grassmann submodel	62
8.1	A Definition of a Higher-Order Grassmann submodel	62
8.2	Conserved Currents	63
8.3	Symmetries of the Higher-Order Grassmann submodel	64

Part I

A Submodel of the Nonlinear CP^1 -Model and Related Submodels in $(1 + 2)$ Dimensions

Chapter 1

A Submodel of the Nonlinear CP^1 -Model

In 1997, Alvarez, Ferreira and Guillen in the interesting paper [AFG1] proposed a new idea to generalize the method in two dimensions. In particular they defined a three dimensional integrability and applied their method to the CP^1 -model in $(1+2)$ dimensions to obtain an infinite number of nontrivial conserved currents. But they have not calculated all forms of conserved currents. In this chapter, we give explicit forms of conserved currents of a submodel of the CP^1 -model and also apply their method to other nonlinear sigma models in $(1+2)$ dimensions to obtain an infinite number of nontrivial conserved currents.

1.1 AFG's Local Integrability Condition in $(1+2)$ Dimensions

Let M be $(1+2)$ -dimensional Minkowski space and $\hat{\mathfrak{g}}$ a Lie algebra. We introduce a connection form A_μ with valued $\hat{\mathfrak{g}}$ and a $\hat{\mathfrak{g}}$ -valued anti-symmetric tensor field $B_{\mu\nu}$ on M . We define the curvature and the covariant derivative

with respect to A_μ .

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad (1.1)$$

$$D_\mu \tilde{B}^\mu \equiv \partial_\mu \tilde{B}^\mu + [A_\mu, \tilde{B}^\mu], \quad (1.2)$$

where

$$\tilde{B}^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\lambda} B_{\nu\lambda} \quad \text{with} \quad \epsilon^{012} = 1 = -\epsilon_{012}. \quad (1.3)$$

We define a local integrability condition according to [AFG1]. If $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{p}$ with \mathfrak{g} a semisimple Lie subalgebra and \mathfrak{p} an abelian ideal of $\hat{\mathfrak{g}}$, then, the local integrability condition is defined as follows:

$$A_\mu \in \mathfrak{g}, \quad B_{\mu\nu} \in \mathfrak{p}, \quad (1.4)$$

$$F_{\mu\nu} = 0 \quad \text{and} \quad D_\mu \tilde{B}^\mu = 0. \quad (1.5)$$

Because of the flatness of A_μ , we can write

$$A_\mu = -\partial_\mu W W^{-1}, \quad (1.6)$$

where W is a function on M having values in the Lie group G corresponding to \mathfrak{g} .

Now we consider the case when $\hat{\mathfrak{g}}$ is given by an abelian extension of \mathfrak{g} . Let R be a representation of \mathfrak{g} , $R : \mathfrak{g} \rightarrow \mathfrak{gl}(P)$, where P is a representation space. The construction of $\hat{\mathfrak{g}}$ is

$$0 \rightarrow P \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0. \quad (1.7)$$

Let $\{T_a\}$ be a basis of \mathfrak{g} and $\{P_i\}$ of P . The commutation relations in $\hat{\mathfrak{g}}$ are

$$\begin{aligned} [T_a, T_b] &= f_{ab}^c T_c, \\ [T_a, P_i] &= P_j R_{ji}(T_a), \\ [P_i, P_j] &= 0, \end{aligned} \quad (1.8)$$

where f_{ab}^c denote the structure constants of \mathfrak{g} and R_{ji} the matrix elements of R . This means that $\hat{\mathfrak{g}}$ is a semi-direct sum of a Lie algebra \mathfrak{g} and an abelian Lie algebra P .

We choose A_μ and $B_{\mu\nu}$ as

$$A_\mu \in \mathfrak{g} \quad \text{and} \quad B_{\mu\nu} \in P \quad (1.9)$$

and we suppose that they satisfy (1.5). Then the current

$$J_\mu \equiv W^{-1} \tilde{B}_\mu W \quad (1.10)$$

is conserved by virtue of the equality

$$\partial_\mu J^\mu = W^{-1} D_\mu \tilde{B}^\mu W = 0, \quad (1.11)$$

where $W^{-1} \tilde{B}_\mu W = \mathcal{R}(W^{-1}) \tilde{B}_\mu$ and $\mathcal{R} : G \rightarrow GL(P)$ such that $d\mathcal{R} = R$.

On the other hand, since $J_\mu \in P$

$$J_\mu = \sum_{i=1}^{\dim P} J_\mu^i P_i, \quad (1.12)$$

$\{J_\mu^i | 1 \leq i \leq \dim P\}$ is a set of conserved currents. Therefore if there are infinitely many different representations R , we get an infinite number of conserved currents.

1.2 AFG's Definition of the CP^1 -submodel

In this section we consider the CP^1 -model in (1+2) dimensions as an effective example of the preceding theory. CP^1 (1-dimensional complex projective space) is identified with $SU(2)/U(1)$ and the embedding $i : CP^1 \rightarrow SU(2)$ is

$$i(CP^1) = \left\{ \frac{1}{\sqrt{1+|v|^2}} \begin{pmatrix} 1 & v \\ -\bar{v} & 1 \end{pmatrix} \mid v \in \mathbb{C} \right\}. \quad (1.13)$$

But according to [AFG1], we set $v = iu$ ($u \in \mathbf{C}$) to obtain

$$i(\mathbf{CP}^1) = \left\{ g(u) = \frac{1}{\sqrt{1+|u|^2}} \begin{pmatrix} 1 & iu \\ i\bar{u} & 1 \end{pmatrix} \mid u \in \mathbf{C} \right\}. \quad (1.14)$$

This form becomes useful later on. We note here that \mathbf{CP}^1 is identified with the projection space

$$\mathbf{CP}^1 = \text{Ad}(SU(2)) \cdot \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}. \quad (1.15)$$

For $g(u) \in i(\mathbf{CP}^1) \subset SU(2)$, we have

$$g(u) \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} g(u)^{-1} = \frac{1}{1+|u|^2} \begin{pmatrix} 1 & -iu \\ i\bar{u} & |u|^2 \end{pmatrix}. \quad (1.16)$$

The action of the \mathbf{CP}^1 -model in $(1+2)$ dimensions is given by

$$\mathcal{A}(u) \equiv \int d^3x \frac{\partial^\mu \bar{u} \partial_\mu u}{(1+|u|^2)^2}, \quad (1.17)$$

where $u : M^{1+2} \rightarrow \mathbf{C}$. Its equation of motion is

$$(1+|u|^2)\partial^\mu \partial_\mu u - 2\bar{u}\partial^\mu u \partial_\mu u = 0. \quad (1.18)$$

This model is invariant under the transformation

$$u \rightarrow \frac{1}{u}. \quad (1.19)$$

It is well-known that this model has three conserved currents (Noether currents)

$$J_\mu^{Noet} = \frac{1}{(1+|u|^2)^2} (\partial_\mu u \bar{u} - u \partial_\mu \bar{u}), \quad (1.20)$$

$$j_\mu = \frac{1}{(1+|u|^2)^2} (\partial_\mu u + u^2 \partial_\mu \bar{u}), \quad (1.21)$$

$$\text{and the complex conjugate } \bar{j}_\mu, \quad (1.22)$$

corresponding to the number of generators of $SU(2)$.

To begin with, we apply the preceding theory to the \mathbf{CP}^1 -model. Let \mathfrak{g} be $\mathfrak{sl}(2, \mathbf{C})$, the Lie algebra of $SL(2, \mathbf{C})$. Let $\{T_+, T_-, T_3\}$ be generators of $\mathfrak{sl}(2, \mathbf{C})$ satisfying

$$[T_3, T_+] = T_+, \quad [T_3, T_-] = -T_-, \quad [T_+, T_-] = 2T_3. \quad (1.23)$$

Usually we choose

$$T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.24)$$

From here we consider a spin j representation of $\mathfrak{sl}(2, \mathbf{C})$. Then $\hat{\mathfrak{g}}$ in (1.8) is given by

$$\begin{aligned} [T_3, P_m^{(j)}] &= mP_m^{(j)}, \\ [T_{\pm}, P_m^{(j)}] &= \sqrt{j(j+1) - m(m \pm 1)} P_{m \pm 1}^{(j)}, \\ [P_m^{(j)}, P_n^{(j)}] &= 0 \end{aligned} \quad (1.25)$$

where $m \in \{-j, -j+1, \dots, j-1, j\}$ and $\{P_m^{(j)} \mid -j \leq m \leq j\}$ is a set of generators of the representation space $P \cong \mathbf{C}^{2j+1}$. We note that $P_j^{(j)}$ ($P_{-j}^{(j)}$) is the highest (lowest) spin state. Now we must choose gauge fields A_μ , $B_{\mu\nu}$ or A_μ , \tilde{B}_μ to satisfy (1.5). From (1.14), we set

$$W = W(u) \equiv \frac{1}{\sqrt{1+|u|^2}} \begin{pmatrix} 1 & iu \\ i\bar{u} & 1 \end{pmatrix} \quad (1.26)$$

and choose

$$\begin{aligned} A_\mu &\equiv -\partial_\mu W W^{-1} \\ &= \frac{-1}{1+|u|^2} \{i\partial_\mu u T_+ + i\partial_\mu \bar{u} T_- + (\partial_\mu u \bar{u} - u \partial_\mu \bar{u}) T_3\}, \end{aligned} \quad (1.27)$$

$$\tilde{B}_\mu^{(1)} \equiv \frac{1}{1+|u|^2} (\partial_\mu u P_1^{(1)} - \partial_\mu \bar{u} P_{-1}^{(1)}). \quad (1.28)$$

Remark 1.1. *The Gauss decomposition of W in (1.26) is given by*

$$W = W_1 \equiv e^{iuT_+} e^{\varphi T_3} e^{i\bar{u}T_-} \quad (1.29)$$

or

$$W = W_2 \equiv e^{i\bar{u}T_-} e^{-\varphi T_3} e^{iuT_+} \quad (1.30)$$

where $\varphi = \log(1 + |u|^2)$.

Now it is easy to show that the \mathbf{CP}^1 -model satisfies the local integrability conditions

$$F_{\mu\nu} = 0 \quad \text{and} \quad D_\mu \tilde{B}^{\mu(1)} = 0 \quad (1.31)$$

from (1.27) and (1.28). Therefore the conserved currents (1.10) are

$$J_\mu^{(1)} \equiv W^{-1} \tilde{B}_\mu^{(1)} W = j_\mu P_1^{(1)} - \sqrt{2} i J_\mu^{Noet} P_0^{(1)} - \bar{j}_\mu P_{-1}^{(1)}, \quad (1.32)$$

where coefficients are given by (1.20), (1.21), (1.22). Next we consider more extended situation than in (1.27), (1.28). That is, we choose

$$\tilde{B}_\mu^{(j)} \equiv \frac{1}{1 + |u|^2} (\partial_\mu u P_1^{(j)} - \partial_\mu \bar{u} P_{-1}^{(j)}) \quad (j = 1, 2, \dots) \quad (1.33)$$

instead of $\tilde{B}_\mu^{(1)}$ in (1.28). In this case $P_1^{(j)}$ ($P_{-1}^{(j)}$) is not the highest (lowest) spin state unless $j = 1$. If we assume $D_\mu \tilde{B}^{\mu(j)} = 0$, where

$$\begin{aligned} D_\mu \tilde{B}^{\mu(j)} = & \frac{1}{(1 + |u|^2)^2} \left\{ \sqrt{j(j+1) - 2} (-i \partial_\mu u \partial^\mu u P_2^{(j)} + i \partial_\mu \bar{u} \partial^\mu \bar{u} P_{-2}^{(j)}) \right. \\ & + \{ (1 + |u|^2) \partial^\mu \partial_\mu u - 2 \bar{u} \partial^\mu u \partial_\mu u \} P_1^{(j)} \\ & \left. + \{ (1 + |u|^2) \partial^\mu \partial_\mu \bar{u} - 2 u \partial^\mu \bar{u} \partial_\mu \bar{u} \} P_{-1}^{(j)} \right\}, \end{aligned} \quad (1.34)$$

we must add a new constraint in addition to the equation of motion (1.18):

$$(1 + |u|^2) \partial^\mu \partial_\mu u - 2 \bar{u} \partial^\mu u \partial_\mu u = 0 \quad \text{and} \quad \partial^\mu u \partial_\mu u = 0.$$

Namely

$$\partial^\mu \partial_\mu u = 0 \quad \text{and} \quad \partial^\mu u \partial_\mu u = 0. \quad (1.35)$$

We call these equations a submodel of the \mathbf{CP}^1 -model (or *the \mathbf{CP}^1 -submodel* for short) the according to [AFG1]. Then the conserved currents are

$$J_\mu^{(j)} \equiv W^{-1} \tilde{B}_\mu^{(j)} W = \sum_{k=-j}^j J_\mu^{(j,k)} P_k^{(j)}. \quad (1.36)$$

In [AFG1] they determined $\{J_\mu^{(j,k)} \mid |k| \leq j\}$ for $j = 1, 2, 3$ only and left the remaining cases. We determine these for any $j \in \mathbf{N}$ in the following section.

1.3 Formulas for a Generalized Potential

We generalize the $\tilde{B}_\mu^{(j)}$ in the preceeding section to the following formula;

$$\tilde{B}_\mu^{(j;m)} = \frac{1}{1 + |u|^2} (\partial_\mu u P_m^{(j)} - \partial_\mu \bar{u} P_{-m}^{(j)}) \quad (1.37)$$

where $m \in \{1, \dots, j\}$. For $m = 1$, $\tilde{B}_\mu^{(j;1)}$ reduces to $\tilde{B}_\mu^{(j)}$ in (1.33). We shall calculate “conserved currents with a parameter m ”:

$$J_\mu^{(j;m)} \equiv W^{-1} \tilde{B}_\mu^{(j;m)} W = \sum_{k=-j}^j J_\mu^{(j;m)}(k) P_k^{(j)}. \quad (1.38)$$

Theorem 1.2. *we have*

(a) *for $k \geq 0$,*

(i) $0 \leq k \leq m$,

$$\begin{aligned} J_\mu^{(j;m)}(k) &= \sqrt{\frac{(j+k)!(j-k)!}{(j+m)!(j-m)!}} \frac{1}{(1+|u|^2)^{j+1}} \\ &\times \left\{ \sum_{n=0}^{j-m} \alpha_n(m, k) |u|^{2n} (-i\bar{u})^{m-k} \partial_\mu u \right. \\ &\quad \left. - (-1)^{j-m} \sum_{n=0}^{j-m} \alpha_{j-m-n}(m, k) |u|^{2n} (-iu)^{m+k} \partial_\mu \bar{u} \right\}, \quad (1.39) \end{aligned}$$

(ii) $m \leq k \leq j$,

$$J_\mu^{(j;m)}(k) = \sqrt{\frac{(j+m)!(j-m)!}{(j+k)!(j-k)!}} \frac{(-iu)^{k-m}}{(1+|u|^2)^{j+1}} \\ \times \left\{ \sum_{n=0}^{j-k} \alpha_n(k, m) |u|^{2n} \partial_\mu u \right. \\ \left. - (-1)^{j-k} \sum_{n=0}^{j-k} \alpha_{j-k-n}(k, m) |u|^{2n} (-iu)^{2m} \partial_\mu \bar{u} \right\}, \quad (1.40)$$

where

$$\alpha_n(m, k) \equiv (-1)^n \binom{j-m}{n} \binom{j+m}{n+m-k}. \quad (1.41)$$

(b) For $k < 0$,

$$J_\mu^{(j;m)}(k) = (-1)^{k+m+1} J_\mu^{(j;m)\dagger}(-k). \quad (1.42)$$

Remark 1.3. We have simple relations:

$$\alpha_n(m, -k) = \frac{(j+m)!(j-m)!}{(j+k)!(j-k)!} \alpha_n(k, -m), \quad (1.43)$$

$$\alpha_{j-m-n}(m, k) = (-1)^{j-m} \alpha_n(m, -k). \quad (1.44)$$

These formulas are also useful in our calculations.

Proof: For simplicity, we put $C_{\pm, m}^{(j)} \equiv \sqrt{j(j+1) - m(m \pm 1)}$. Then, we have

$$C_{+, m}^{(j)} = C_{-, m+1}^{(j)} = C_{+, -m-1}^{(j)}, \quad C_{-, m}^{(j)} = C_{-, -m+1}^{(j)}, \quad (1.45)$$

$$\prod_{a=1}^s C_{+, a-1+m}^{(j)} = \sqrt{\frac{(j+s+m)!(j-m)!}{(j-s-m)!(j+m)!}}, \quad (1.46)$$

$$\prod_{b=1}^l C_{+, -b+s+m}^{(j)} = \sqrt{\frac{(j+s+m)!(j-s-m+l)!}{(j-s-m)!(j+s+m-l)!}}. \quad (1.47)$$

By using two-type of the Gauss decomposition (1.29), (1.30)

$$W = W_1 = W_2$$

and a formula

$$\mathcal{R}(e^X)Y = \exp(\mathcal{R}(X))Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \cdots, \quad (1.48)$$

we have

$$\begin{aligned} J_\mu^{(j;m)} &= \mathcal{R}(W^{-1})\tilde{B}_\mu^{(j;m)} \\ &= \frac{\partial_\mu u}{1 + |u|^2} \mathcal{R}(W_1^{-1})P_m^{(j)} - \frac{\partial_\mu \bar{u}}{1 + |u|^2} \mathcal{R}(W_2^{-1})P_{-m}^{(j)} \\ &= \sum_{s=0}^{j-m} \sum_{l=0}^{j+s+m} \frac{1}{s!l!} \frac{1}{(1 + |u|^2)^{s+m+1}} \prod_{a=1}^s C_{+,a-1+m}^{(j)} \prod_{b=1}^l C_{+,-b+s+m}^{(j)} \\ &\quad \times \{(-iu)^s (-i\bar{u})^l \partial_\mu u P_{s-l+m}^{(j)} - (-i\bar{u})^s (-iu)^l \partial_\mu \bar{u} P_{-(s-l+m)}^{(j)}\} \\ &= \sum_{s=0}^{j-m} \sum_{l=0}^{j+s+m} \frac{1}{s!l!} \frac{1}{(1 + |u|^2)^{s+m+1}} \frac{(j+s+m)!}{(j-s-m)!} \sqrt{\frac{(j-m)!(j-s-m+l)!}{(j+m)!(j+s+m-l)!}} \\ &\quad \times \{(-iu)^s (-i\bar{u})^l \partial_\mu u P_{s-l+m}^{(j)} - (-i\bar{u})^s (-iu)^l \partial_\mu \bar{u} P_{-(s-l+m)}^{(j)}\}. \end{aligned}$$

We put $k = s - l + m$ and calculate $J_\mu^{(j;m)}(k)$.

(i) In the case of $0 \leq k \leq m$, we note

$$(-iu)^s (-i\bar{u})^l = (-|u|^2)^s (-i\bar{u})^{m-k} \quad (1.49)$$

and

$$\frac{1}{(1 + |u|^2)^{s+m+1}} = \frac{1}{(1 + |u|^2)^{j+1}} \sum_{t=0}^{j-s-m} \frac{(j-s-m)!}{t!(j-s-m-t)!} |u|^{2j-2s-2m-2t}. \quad (1.50)$$

If we put $n = j - m - t$, we find that the coefficient of $\frac{|u|^{2n}(-i\bar{u})^{m-k}\partial_\mu u}{(1+|u|^2)^{j+1}}$ is

$$\begin{aligned} & \sqrt{\frac{(j-m)!(j-k)!}{(j+m)!(j+k)!}} \sum_{s=0}^n (-1)^s \frac{(j+s+m)!}{s!(s+m-k)!(j-m-n)!(n-s)!} \\ &= \sqrt{\frac{(j+k)!(j-k)!}{(j+m)!(j-m)!}} \alpha_n(m, k). \end{aligned}$$

Similarly, we also find that the coefficient of $\frac{|u|^{2n}(-iu)^{m+k}\partial_\mu \bar{u}}{(1+|u|^2)^{j+1}}$ is

$$-\sqrt{\frac{(j+k)!(j-k)!}{(j+m)!(j-m)!}} \alpha_n(m, -k).$$

Therefore we obtain (1.39).

(ii) In the case of $m \leq k \leq j$,

$$(-iu)^s(-i\bar{u})^l = (-|u|^2)^l(-iu)^{k-m}. \quad (1.51)$$

If we exchange m and k in the proof of (i), then we can calculate (1.40) in a similar way.

(iii) By using

$$\tilde{B}_\mu^{(j;m)\dagger} = (-1)^{m+1} \tilde{B}_\mu^{(j;m)} \quad (1.52)$$

and

$$W^\dagger = W^{-1}, \quad (1.53)$$

we have

$$J_\mu^{(j;m)\dagger} = (-1)^{m+1} J_\mu^{(j;m)}. \quad (1.54)$$

(1.54) implies the relation (1.42). \square

1.4 Conserved Currents associated with Spin j Representation of $SU(2)$

In this section, we describe the conserved currents (1.36) of the CP^1 -submodel. We set $m = 1$ in theorem 1.2.

Proposition 1.4. *For any $j \in \mathbf{N}$, conserved currents of the CP^1 -submodel associated with spin j representation of $SU(2)$ are given by as follows:*

(a) for $k \geq 0$,

(i) $k = 0$,

$$J_\mu^{(j,0)} = J_\mu^{(j;1)}(0) = \sqrt{\frac{j}{j+1}} \frac{-i}{(1+|u|^2)^{j+1}} (\bar{u} \partial_\mu u - u \partial_\mu \bar{u}) \\ \times \sum_{n=0}^{j-1} (-1)^n \binom{j-1}{n} \binom{j+1}{n+1} |u|^{2n}, \quad (1.55)$$

(ii) $1 \leq k \leq j$,

$$J_\mu^{(j,k)} = J_\mu^{(j;1)}(k) = \sqrt{\frac{(j+1)!(j-1)!}{(j+k)!(j-k)!}} \frac{(-iu)^{k-1}}{(1+|u|^2)^{j+1}} \\ \times \left\{ \sum_{n=0}^{j-k} (-1)^n \binom{j-k}{n} \binom{j+k}{n+k-1} |u|^{2n} \partial_\mu u \right. \\ \left. + \sum_{n=0}^{j-k} (-1)^n \binom{j-k}{n} \binom{j+k}{n+k+1} |u|^{2n} u^2 \partial_\mu \bar{u} \right\}. \quad (1.56)$$

(b) For $k < 0$,

$$J_\mu^{(j,k)} = J_\mu^{(j;1)}(k) = (-1)^k J_\mu^{(j;1)\dagger}(-k). \quad (1.57)$$

Remark 1.5. *By a little calculation, we can express proposition 1.4 as follows (see [FS1]):*

(a) for $j \geq m \geq 1$,

$$J_\mu^{(j,m)} = \sqrt{\frac{(j+m)!}{j(j+1)(j-m)!}} \frac{(-iu)^{m-1}}{(1+|u|^2)^{j+1}} \\ \times \left(\sum_{n=0}^{j-m} \alpha_n^{(j,m)} |u|^{2n} \partial_\mu u + (-1)^{j-m} \sum_{n=0}^{j-m} \alpha_{j-m-n}^{(j,m)} |u|^{2n} u^2 \partial_\mu \bar{u} \right), \quad (1.58)$$

(b) for $m = 0$,

$$J_\mu^{(j,0)} = -i\sqrt{j(j+1)} \frac{(\bar{u}\partial_\mu u - u\partial_\mu \bar{u})}{(1+|u|^2)^{j+1}} \sum_{n=0}^{j-1} \gamma_n^{(j,0)} |u|^{2n}, \quad (1.59)$$

(c) for $j \geq m \geq 1$,

$$J_\mu^{(j,-m)} = (-1)^m J_\mu^{(j,m)\dagger}, \quad (1.60)$$

where coefficients are

$$\alpha_n^{(j,m)} = (-1)^n \frac{n!}{(m+n-1)!} \binom{j-m}{n} \binom{j+1}{n}, \quad (1.61)$$

$$\gamma_n^{(j,0)} = (-1)^n \frac{1}{j} \binom{j}{n} \binom{j}{n+1}. \quad (1.62)$$

Chapter 2

Submodels of Related Models

2.1 CP^1 -like Models and their Submodels

In this section, we consider the CP^1 -like sigma models in $(1+2)$ dimensions. First of all, we fix $j \in \mathbf{N}$. The action of such a model is given by

$$\mathcal{A}_j(u) \equiv \int d^3x \frac{\partial^\mu \bar{u} \partial_\mu u}{(1 + |u|^2)^{j+1}}, \quad (2.1)$$

where $u : M^{1+2} \rightarrow \mathbf{C}$. When $j = 1$, (2.1) reduces to the CP^1 -model.

Remark 2.1. *Since the action (2.1) is not invariant under the transformation $u \rightarrow 1/u$ in (1.19), alternatively, we may consider an invariant action*

$$\tilde{\mathcal{A}}_j(u) \equiv \int d^3x \frac{(1 + |u|^{2(j-1)}) \partial^\mu \bar{u} \partial_\mu u}{(1 + |u|^2)^{j+1}}. \quad (2.2)$$

But for the sake of simplicity, we consider (2.1) only in this thesis.

The equation of motion of (2.1) reads

$$(1 + |u|^2) \partial^\mu \partial_\mu u - (j + 1) \bar{u} \partial^\mu u \partial_\mu u = 0. \quad (2.3)$$

Taking an analogy of section 1.2, we set A_μ the same as (1.27) and \tilde{B}_μ as

$$\tilde{B}_\mu = \frac{1}{1 + |u|^2} (\partial_\mu u P_j^{(j)} - \partial_\mu \bar{u} P_{-j}^{(j)}) \quad (2.4)$$

where $P_j^{(j)}$ ($P_{-j}^{(j)}$) is the highest (lowest) spin state. Let us calculate $D_\mu \tilde{B}^\mu$.

$$\begin{aligned} D_\mu \tilde{B}^\mu &= -i\sqrt{2j} \frac{\partial^\mu \bar{u} \partial_\mu u}{(1+|u|^2)^2} P_{j-1}^{(j)} + i\sqrt{2j} \frac{\partial^\mu \bar{u} \partial_\mu u}{(1+|u|^2)^2} P_{-j+1}^{(j)} \\ &+ \frac{(1+|u|^2) \partial^\mu \partial_\mu u - (j+1) \bar{u} \partial^\mu u \partial_\mu u + (j-1) u \partial^\mu \bar{u} \partial_\mu u}{(1+|u|^2)^2} P_{-j}^{(j)} \\ &+ \frac{(1+|u|^2) \partial^\mu \partial_\mu \bar{u} - (j+1) u \partial^\mu \bar{u} \partial_\mu \bar{u} + (j-1) \bar{u} \partial^\mu u \partial_\mu u}{(1+|u|^2)^2} P_j^{(j)}. \end{aligned} \quad (2.5)$$

If we assume $D_\mu \tilde{B}^\mu = 0$, then we have

$$(1+|u|^2) \partial^\mu \partial_\mu u - (j+1) \bar{u} \partial^\mu u \partial_\mu u = 0 \quad \text{and} \quad \partial^\mu \bar{u} \partial_\mu u = 0. \quad (2.6)$$

We call these equations a submodel of the \mathbf{CP}^1 -like model. For this model, the local integrability conditions ($F_{\mu\nu} = 0$ and $D_\mu \tilde{B}^\mu = 0$) are satisfied. Therefore, the conserved currents are

$$J_\mu \equiv W^{-1} \tilde{B}_\mu W = \sum_{k=-j}^j J_\mu^{(j;j)}(k) P_k^{(j)}. \quad (2.7)$$

We set $m = j$ in theorem 1.2 to obtain the conserved currents (2.7) of the submodel of the \mathbf{CP}^1 -like model.

Proposition 2.2. *We have*

(a) for $0 \leq k \leq j$,

$$\begin{aligned} J_\mu^{(j;j)}(k) &= \sqrt{\frac{(2j)!}{(j+k)!(j-k)!}} \frac{1}{(1+|u|^2)^{j+1}} \\ &\times \{ (-i\bar{u})^{j-k} \partial_\mu u - (-iu)^{j+k} \partial_\mu \bar{u} \}. \end{aligned} \quad (2.8)$$

(b) For $k < 0$,

$$J_\mu^{(j;j)}(k) = (-1)^{j+1+k} J_\mu^{(j;j)\dagger}(-k). \quad (2.9)$$

Moreover, we can remove the constraint $j \geq |k|$. Namely, we have

Corollary 2.3.

$$J_\mu^{(n)} = \frac{\bar{u}^n \partial_\mu u}{(1 + |u|^2)^{j+1}}, \quad n \in \mathbf{Z} \quad (2.10)$$

and its complex conjugate $\bar{J}_\mu^{(n)}$ are conserved currents of the submodel of the \mathbf{CP}^1 -like model.

The proof is a direct calculation.

2.2 QP^1 -model and its submodel

In this section, we consider the QP^1 -model in $(1 + 2)$ dimensions. We set

$$QP^1 \equiv \{\phi \in U(1, 1) | \phi^\dagger = \phi\}_* , \quad (2.11)$$

where $\{\cdots\}_*$ means a connected component containing the unit matrix. QP^1 (1-dimensional quasi projective space) is identified with $SU(1, 1)/U(1) \cong D = \{z \in \mathbf{C} | |z| < 1\}$ and the embedding $i : QP^1 \rightarrow SU(1, 1)$ is

$$i(QP^1) = \left\{ \frac{1}{\sqrt{1 - |u|^2}} \begin{pmatrix} 1 & iu \\ -i\bar{u} & 1 \end{pmatrix} | u \in D \right\}. \quad (2.12)$$

Here D is the Poincare disk. We note here that QP^1 is identified with the quasi projection space

$$QP^1 \cong \left\{ \frac{1}{1 - |u|^2} \begin{pmatrix} 1 & -iu \\ -i\bar{u} & -|u|^2 \end{pmatrix} | u \in D \right\}. \quad (2.13)$$

See [Fuj], in detail.

The action of the QP^1 -model in $(1 + 2)$ dimensions is given by

$$\mathcal{A}(u) \equiv \int d^3x \frac{\partial^\mu \bar{u} \partial_\mu u}{(1 - |u|^2)^2}, \quad (2.14)$$

where $u : M^{1+2} \rightarrow D$. Its equation of motion is

$$(1 - |u|^2)\partial^\mu \partial_\mu u + 2\bar{u}\partial^\mu u \partial_\mu u = 0. \quad (2.15)$$

This model is invariant under the transformation $u \rightarrow 1/u$ in (1.19). This model has three conserved currents

$$J_\mu^{Noet} = \frac{1}{(1 - |u|^2)^2} (\partial_\mu u \bar{u} - u \partial_\mu \bar{u}), \quad (2.16)$$

$$j_\mu = \frac{1}{(1 - |u|^2)^2} (\partial_\mu u - u^2 \partial_\mu \bar{u}), \quad (2.17)$$

$$\text{and the complex conjugate } \bar{j}_\mu, \quad (2.18)$$

corresponding to the number of generators of $SU(1, 1)$. The complexification of both $SU(2)$ in section 1.2 and $SU(1, 1)$ in this section is just $SL(2, \mathbf{C})$. Therefore, the arguments in section 1.2 are still valid in this section. Namely we set

$$W \equiv W(u) = \frac{1}{\sqrt{1 - |u|^2}} \begin{pmatrix} 1 & iu \\ -i\bar{u} & 1 \end{pmatrix}. \quad (2.19)$$

For this, the Gauss decomposition is given by

$$W = W_1 \equiv e^{iuT_+} e^{\varphi T_3} e^{-i\bar{u}T_-} \quad (2.20)$$

or

$$W = W_2 \equiv e^{-i\bar{u}T_-} e^{-\varphi T_3} e^{iuT_+} \quad (2.21)$$

where $\varphi = \log(1 - |u|^2)$. We choose A_μ and $\tilde{B}_\mu^{(1)}$ as

$$\begin{aligned} A_\mu &\equiv -\partial_\mu W W^{-1} \\ &= \frac{1}{1 + |u|^2} \{-i\partial_\mu u T_+ + i\partial_\mu \bar{u} T_- + (\partial_\mu u \bar{u} - u \partial_\mu \bar{u}) T_3\}, \end{aligned} \quad (2.22)$$

$$\tilde{B}_\mu^{(1)} \equiv \frac{1}{1 - |u|^2} (\partial_\mu u P_1^{(1)} + \partial_\mu \bar{u} P_{-1}^{(1)}). \quad (2.23)$$

Then, we easily have

$$F_{\mu\nu} = 0 \quad \text{and} \quad D_\mu \tilde{B}^{\mu(1)} = 0, \quad (2.24)$$

so the conserved currents are

$$J_\mu^{(1)} \equiv W^{-1} \tilde{B}_\mu^{(1)} W = j_\mu P_1^{(1)} + \sqrt{2} i J_\mu^{Noet} P_0^{(1)} + \bar{j}_\mu P_{-1}^{(1)}, \quad (2.25)$$

where coefficients are (2.16), (2.17), (2.18).

Next we consider the extended situation as shown in section 1.2. We choose

$$\tilde{B}_\mu^{(j)} \equiv \frac{1}{1 - |u|^2} (\partial_\mu u P_1^{(j)} + \partial_\mu \bar{u} P_{-1}^{(j)}) \quad (2.26)$$

instead of $\tilde{B}_\mu^{(1)}$ in (2.23). Then

$$\begin{aligned} D_\mu \tilde{B}^{\mu(j)} &= \frac{1}{(1 - |u|^2)^2} \left\{ \sqrt{j(j+1) - 2} (-i \partial_\mu u \partial^\mu u P_2^{(j)} + i \partial_\mu \bar{u} \partial^\mu \bar{u} P_{-2}^{(j)}) \right. \\ &\quad + \{(1 - |u|^2) \partial^\mu \partial_\mu u + 2 \bar{u} \partial^\mu u \partial_\mu u\} P_1^{(j)} \\ &\quad \left. + \{(1 - |u|^2) \partial^\mu \partial_\mu \bar{u} + 2 u \partial^\mu \bar{u} \partial_\mu \bar{u}\} P_{-1}^{(j)} \right\}. \end{aligned} \quad (2.27)$$

Therefore, we consider a submodel of the QP^1 -model,

$$\partial^\mu \partial_\mu u = 0 \quad \text{and} \quad \partial^\mu u \partial_\mu u = 0. \quad (2.28)$$

Then we have the local integrability conditions $F_{\mu\nu} = 0$ and $D_\mu \tilde{B}^{\mu(j)} = 0$.

The conserved currents are

$$J_\mu^{(j)} \equiv W^{-1} \tilde{B}_\mu^{(j)} W = \sum_{k=-j}^j J_\mu^{(j,k)} P_k^{(j)}. \quad (2.29)$$

For this case, we can use proposition 1.4. Namely, we restrict u to $|u| < 1$ and replace

$$u \rightarrow u, \quad \bar{u} \rightarrow -\bar{u} \quad (2.30)$$

in proposition 1.4.

Proposition 2.4. *For any $j \in \mathbf{N}$, conserved currents of the submodel of the QP^1 -model associated with spin j representation of $SU(1,1)$ are given by as follows:*

(a) for $k \geq 0$,

(i) $k = 0$,

$$J_\mu^{(j,0)} = J_\mu^{(j;1)}(0) = \sqrt{\frac{j}{j+1}} \frac{i}{(1-|u|^2)^{j+1}} (\bar{u} \partial_\mu u - u \partial_\mu \bar{u}) \times \sum_{n=0}^{j-1} \binom{j-1}{n} \binom{j+1}{n+1} |u|^{2n}, \quad (2.31)$$

(ii) $1 \leq k \leq j$,

$$J_\mu^{(j,k)} = J_\mu^{(j;1)}(k) = \sqrt{\frac{(j+1)!(j-1)!}{(j+k)!(j-k)!}} \frac{(-iu)^{k-1}}{(1-|u|^2)^{j+1}} \times \left\{ \sum_{n=0}^{j-k} \binom{j-k}{n} \binom{j+k}{n+k-1} |u|^{2n} \partial_\mu u - \sum_{n=0}^{j-k} \binom{j-k}{n} \binom{j+k}{n+k+1} |u|^{2n} u^2 \partial_\mu \bar{u} \right\} \quad (2.32)$$

(b) For $k < 0$,

$$J_\mu^{(j,k)} = J_\mu^{(j;1)}(k) = J_\mu^{(j;1)\dagger}(-k). \quad (2.33)$$

Remark 2.5. *By a little calculation, we can express proposition 2.4 as follows (see [FS1]):*

(a) for $j \geq m \geq 1$,

$$J_\mu^{(j,m)} = \sqrt{\frac{(j+m)!}{j(j+1)(j-m)!}} \frac{(-iu)^{m-1}}{(1-|u|^2)^{j+1}} \times \left(\sum_{n=0}^{j-m} \tilde{\alpha}_n^{(j,m)} |u|^{2n} \partial_\mu u - \sum_{n=0}^{j-m} \tilde{\alpha}_{j-m-n}^{(j,m)} |u|^{2n} u^2 \partial_\mu \bar{u} \right), \quad (2.34)$$

(b) for $m = 0$,

$$J_{\mu}^{(j,0)} = i\sqrt{j(j+1)} \frac{(\bar{u}\partial_{\mu}u - u\partial_{\mu}\bar{u})}{(1-|u|^2)^{j+1}} \sum_{n=0}^{j-1} \tilde{\gamma}_n^{(j,0)} |u|^{2n}, \quad (2.35)$$

(c) for $j \geq m \geq 1$,

$$J_{\mu}^{(j,-m)} = J_{\mu}^{(j,m)\dagger}, \quad (2.36)$$

where coefficients are

$$\tilde{\alpha}_n^{(j,m)} = \frac{n!}{(m+n-1)!} \binom{j-m}{n} \binom{j+1}{n}, \quad (2.37)$$

$$\tilde{\gamma}_n^{(j,0)} = \frac{1}{j} \binom{j}{n} \binom{j}{n+1}. \quad (2.38)$$

Part II

A Submodel of the Nonlinear Grassmann Model in Any Dimension

Chapter 3

A Definition of a Submodel of the Nonlinear Grassmann Model

3.1 Grassmann Manifolds (Projection-Matrix Expression)

Let $M(m, n; \mathbf{C})$ be the set of $m \times n$ matrices over \mathbf{C} and we write $M(n; \mathbf{C}) \equiv M(n, n; \mathbf{C})$ for simplicity. For a pair (j, N) with $1 \leq j \leq N - 1$, we set I , O as a unit matrix, a zero matrix in $M(j; \mathbf{C})$ and I' , O' as ones in $M(N - j; \mathbf{C})$ respectively.

We define a Grassmann manifold for the pair (j, N) above as a set of projection matrices:

$$G_{j,N}(\mathbf{C}) \equiv \{P \in M(N; \mathbf{C}) | P^2 = P, P^\dagger = P, \text{tr} P = j\}. \quad (3.1)$$

Then

$$G_{j,N}(\mathbf{C}) = \left\{ U \begin{pmatrix} I & \\ & O' \end{pmatrix} U^\dagger | U \in U(N) \right\} \quad (3.2)$$

$$\cong \frac{U(N)}{U(j) \times U(N - j)}. \quad (3.3)$$

In the case of $j = 1$, we usually write $G_{1,N}(\mathbf{C}) = \mathbf{C}P^{N-1}$.

Next, we introduce a local chart for $G_{j,N}(\mathbf{C})$. For $Z \in M(N - j, j; \mathbf{C})$, an element of a neighborhood of $\begin{pmatrix} I & \\ & O' \end{pmatrix}$ in $G_{j,N}(\mathbf{C})$ is expressed as

$$P_0(Z) = \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix} \begin{pmatrix} I & \\ & O' \end{pmatrix} \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix}^{-1}. \quad (3.4)$$

Since

$$\begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix}^{-1} = \begin{pmatrix} (I + Z^\dagger Z)^{-1} & 0 \\ 0 & (I' + ZZ^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} I & Z^\dagger \\ -Z & I' \end{pmatrix}, \quad (3.5)$$

this is also written as

$$P_0(Z) = \begin{pmatrix} (I + Z^\dagger Z)^{-1} & (I + Z^\dagger Z)^{-1} Z^\dagger \\ Z(I + Z^\dagger Z)^{-1} & Z(I + Z^\dagger Z)^{-1} Z^\dagger \end{pmatrix}. \quad (3.6)$$

We note here relations

$$Z(I + Z^\dagger Z)^{-1} = (I' + ZZ^\dagger)^{-1} Z, \quad (3.7)$$

$$(I' + ZZ^\dagger)^{-1} = I' - Z(I + Z^\dagger Z)^{-1} Z^\dagger. \quad (3.8)$$

Since any P in $G_{j,N}(\mathbf{C})$ is written as $P = U \begin{pmatrix} I & \\ & O' \end{pmatrix} U^\dagger$ for some $U \in U(N)$ by (3.2), an element of a neighborhood of P in $G_{j,N}(\mathbf{C})$ is expressed as

$$P(Z) = U P_0(Z) U^\dagger. \quad (3.9)$$

Now, we prepare useful lemmas.

Lemma 3.1. *For $g = g(Z) \in GL(N; \mathbf{C})$ and $Y = Y(Z) \in M(N; \mathbf{C})$, we have*

$$d(gYg^{-1}) = g(dY + [g^{-1}dg, Y])g^{-1}. \quad (3.10)$$

The proof is a direct calculation.

Lemma 3.2.

$$\begin{aligned}
(i) \quad dP_0 &= \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix} \begin{pmatrix} (I + Z^\dagger Z)^{-1} dZ^\dagger & \\ (I' + ZZ^\dagger)^{-1} dZ & \end{pmatrix} \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix}^{-1} \\
& \quad (3.11)
\end{aligned}$$

$$= \begin{pmatrix} I & Z^\dagger \\ -Z & I' \end{pmatrix}^{-1} \begin{pmatrix} dZ & dZ^\dagger \end{pmatrix} \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix}^{-1}, \quad (3.12)$$

$$(ii) \quad [P_0, dP_0] = \begin{pmatrix} I & Z^\dagger \\ -Z & I' \end{pmatrix}^{-1} \begin{pmatrix} -dZ & dZ^\dagger \end{pmatrix} \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix}^{-1}. \quad (3.13)$$

proof: (i) We put

$$g \equiv \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix}, \quad E_0 \equiv \begin{pmatrix} I & \\ & O' \end{pmatrix}, \quad a \equiv g^\dagger g = gg^\dagger \quad (3.14)$$

for simplicity. We note that

$$g^{-1} = g^\dagger a^{-1} = a^{-1} g^\dagger, \quad dg^\dagger = -dg. \quad (3.15)$$

Then, by using lemma 3.1 and

$$g^{-1} dg = \begin{pmatrix} (I + Z^\dagger Z)^{-1} Z^\dagger dZ & -(I + Z^\dagger Z)^{-1} dZ^\dagger \\ (I' + ZZ^\dagger)^{-1} dZ & Z(I + Z^\dagger Z)^{-1} dZ^\dagger \end{pmatrix}, \quad (3.16)$$

we have

$$\begin{aligned}
dP_0 &= d(gE_0g^{-1}) \\
&= g [g^{-1}dg, E_0] g^{-1} \\
&= g \begin{pmatrix} (I + Z^\dagger Z)^{-1} dZ^\dagger & \\ (I' + ZZ^\dagger)^{-1} dZ & \end{pmatrix} g^{-1} \\
&= ga^{-1} \begin{pmatrix} dZ & dZ^\dagger \end{pmatrix} g^{-1} \\
&= (g^\dagger)^{-1} \begin{pmatrix} dZ & dZ^\dagger \end{pmatrix} g^{-1}.
\end{aligned}$$

(ii) Since $[E_0, a^{-1}] = 0$,

$$\begin{aligned}
[P_0, dP_0] &= [gE_0g^{-1}, ga^{-1} \begin{pmatrix} & dZ^\dagger \\ dZ & \end{pmatrix} g^{-1}] \\
&= g [E_0, a^{-1} \begin{pmatrix} & dZ^\dagger \\ dZ & \end{pmatrix}] g^{-1} \\
&= ga^{-1} [E_0, \begin{pmatrix} & dZ^\dagger \\ dZ & \end{pmatrix}] g^{-1} \\
&= (g^\dagger)^{-1} \begin{pmatrix} & dZ^\dagger \\ -dZ & \end{pmatrix} g^{-1}. \quad \square
\end{aligned}$$

3.2 The Nonlinear Grassmann Sigma Model

Let M^{1+n} be a $(1+n)$ -dimensional Minkowski space ($n \in \mathbf{N}$) with a metric $\eta = (\eta_{\mu\nu}) = \text{diag}(1, -1, \dots, -1)$. For fixed (j, N) , the nonlinear Grassmann sigma model in any dimension is defined by the following action:

$$\mathcal{A}(P) \equiv \frac{1}{2} \int d^{1+n}x \, \text{tr} \partial_\mu P \partial^\mu P, \quad (3.17)$$

where

$$P : M^{1+n} \longrightarrow G_{j,N}(\mathbf{C}).$$

Its equations of motion read

$$[P, \square P] \equiv [P, \partial_\mu \partial^\mu P] = 0. \quad (3.18)$$

Since

$$0 = [P, \partial_\mu \partial^\mu P] = \partial^\mu [P, \partial_\mu P], \quad (3.19)$$

$[P, \partial_\mu P]$ are conserved currents. In fact, they are the Noether currents corresponding to $U(N)$ -symmetry.

Next, we express the action by using the local coordinate introducing in section 3.1. By (3.9), we can put

$$P(Z) = U \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix} \begin{pmatrix} I & \\ & O' \end{pmatrix} \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix}^{-1} U^\dagger, \quad (3.20)$$

where

$$Z : M^{1+n} \longrightarrow M(N - j, j; \mathbf{C})$$

and U is a constant unitary matrix. Then,

$$\partial_\mu P(Z) = U \begin{pmatrix} I & Z^\dagger \\ -Z & I' \end{pmatrix}^{-1} \begin{pmatrix} & \partial_\mu Z^\dagger \\ \partial_\mu Z & \end{pmatrix} \begin{pmatrix} I & -Z^\dagger \\ Z & I' \end{pmatrix}^{-1} U^\dagger \quad (3.21)$$

by (3.12).

Lemma 3.3. *We have*

(i) *the action*

$$\mathcal{A}(Z) = \int d^{1+n}x \operatorname{tr}(I + Z^\dagger Z)^{-1} \partial^\mu Z^\dagger (I' + Z Z^\dagger)^{-1} \partial_\mu Z, \quad (3.22)$$

(ii) *the equations of motion*

$$\partial^\mu \partial_\mu Z - 2 \partial^\mu Z (I + Z^\dagger Z)^{-1} Z^\dagger \partial_\mu Z = 0. \quad (3.23)$$

proof: (i) By (3.11) and (3.17), we obtain (3.22).

(ii) In the following, we put $U = 1$ in (3.20) without loss of generality. By using (3.15), we have

$$\begin{aligned} [P, \square P] &= \partial^\mu [P, \partial_\mu P] \\ &= \partial^\mu ((g^\dagger)^{-1} (\partial_\mu g^\dagger) g^{-1}) \\ &= (g^\dagger)^{-1} \{ \partial^\mu \partial_\mu g^\dagger - (\partial^\mu g^\dagger) (g^\dagger)^{-1} \partial_\mu g^\dagger - \partial_\mu g^\dagger g^{-1} \partial^\mu g \} g^{-1} \\ &= (g^\dagger)^{-1} \{ \partial^\mu \partial_\mu g^\dagger - \partial^\mu g ((g^\dagger)^{-1} - g^{-1}) \partial_\mu g \} g^{-1} \\ &= (g^\dagger)^{-1} \{ \partial^\mu \partial_\mu g^\dagger - \partial^\mu g a^{-1} (g - g^\dagger) \partial_\mu g \} g^{-1} = 0 \end{aligned}$$

(3.23) and its complex conjugate follow by this equation. \square

Let us consider the case of $j = 1$ (the \mathbf{CP}^{N-1} -model). If we set $Z = \mathbf{u} = (u_1, \dots, u_{N-1})^t$ where $u_i : M^{1+n} \longrightarrow \mathbf{C}$ and remark that

$$1 + \mathbf{u}^\dagger \mathbf{u} = 1 + \sum_{i=1}^{N-1} |u_i|^2,$$

$$(I' + \mathbf{u} \mathbf{u}^\dagger)^{-1} = I' - \frac{\mathbf{u} \mathbf{u}^\dagger}{1 + \mathbf{u}^\dagger \mathbf{u}}$$

from (3.8),

Corollary 3.4. *we have*

(i) *the action*

$$\mathcal{A}(\mathbf{u}) = \int d^{1+n}x \frac{(1 + \mathbf{u}^\dagger \mathbf{u}) \partial^\mu \mathbf{u}^\dagger \partial_\mu \mathbf{u} - \partial^\mu \mathbf{u}^\dagger \mathbf{u} \partial_\mu \mathbf{u}}{(1 + \mathbf{u}^\dagger \mathbf{u})^2}, \quad (3.24)$$

(ii) *the equations of motion*

$$(1 + \mathbf{u}^\dagger \mathbf{u}) \partial^\mu \partial_\mu \mathbf{u} - 2 \mathbf{u}^\dagger \partial_\mu \mathbf{u} \partial^\mu \mathbf{u} = 0. \quad (3.25)$$

In particular, in the case of $N = 2$ (the \mathbf{CP}^1 -model),

Corollary 3.5. *we have*

(i) *the action*

$$\mathcal{A}(u) = \int d^{1+n}x \frac{\partial^\mu \bar{u} \partial_\mu u}{(1 + |u|^2)^2}, \quad (3.26)$$

(ii) *the equations of motion*

$$(1 + |u|^2) \partial^\mu \partial_\mu u - 2 \bar{u} \partial_\mu u \partial^\mu u = 0. \quad (3.27)$$

Remark 3.6. *We compare formulations of the nonlinear \mathbf{CP}^1 -model in AFG's theory and ours briefly. In \mathbf{CP}^1 -case, we take g as*

$$g = \frac{1}{\sqrt{1+|u|^2}} \begin{pmatrix} 1 & -u \\ \bar{u} & 1 \end{pmatrix} \in SU(2), \quad (3.28)$$

and we put

$$P = gE_0g^{-1}, \quad \text{where } E_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

As well as the proof of lemma 3.2, we have

$$\partial_\mu P = g [g^{-1}\partial_\mu g, E_0] g^{-1} \quad (3.29)$$

and

$$[P, \partial_\mu P] = g [E_0, [g^{-1}\partial_\mu g, E_0]] g^{-1}. \quad (3.30)$$

Then, we observe a quantity

$$\begin{aligned} [E_0, [g^{-1}\partial_\mu g, E_0]] &= \frac{1}{1+|u|^2} \begin{pmatrix} 0 & \partial_\mu u \\ -\partial_\mu \bar{u} & 0 \end{pmatrix} \\ &= \frac{1}{1+|u|^2} (\partial_\mu u T_+ - \partial_\mu \bar{u} T_-). \end{aligned} \quad (3.31)$$

This corresponds with

$$\tilde{B}_\mu^{(1)} = \frac{1}{1+|u|^2} (\partial_\mu u P_1^{(1)} - \partial_\mu \bar{u} P_{-1}^{(1)}) \quad (3.32)$$

in section 1.2, because both T_+ and $P_1^{(1)}$ (resp. T_- and $P_{-1}^{(1)}$) are highest (resp. lowest) weight vector of adjoint representation (i.e. spin 1-representaion) of $\mathfrak{su}(2)$. Therefore, a correspondence of the Noether currents is

$$J_\mu \equiv W^{-1} \tilde{B}_\mu^{(1)} W \leftrightarrow [P, \partial_\mu P] = g [E_0, [g^{-1}\partial_\mu g, E_0]] g^{-1}. \quad (3.33)$$

Moreover, (3.33) implies a correspondence of expressions of the \mathbf{CP}^1 -model.

$$\partial^\mu J_\mu = W^{-1} D^\mu \tilde{B}_\mu^{(1)} W = 0 \leftrightarrow \partial^\mu [P, \partial_\mu P] = [P, \square P] = 0. \quad (3.34)$$

3.3 A Definition of a Submodel

In this section, we define a submodel of the nonlinear Grassmann model. Let us remind equations of motion of the $G_{j,N}(\mathbf{C})$ -model

$$[P, \square P] = 0.$$

For $P \in G_{j,N}(\mathbf{C})$, the tensor product $P \otimes P$ of P is an element of $G_{j^2,N^2}(\mathbf{C})$. Therefore, we assume that $P \otimes P$ is also the solution of the $G_{j^2,N^2}(\mathbf{C})$ -model, namely

$$[P \otimes P, \square(P \otimes P)] = 0. \quad (3.35)$$

Transforming this, we have

$$[P, \square P] \otimes P + P \otimes [P, \square P] + [P, \partial_\mu P] \otimes \partial^\mu P + \partial^\mu P \otimes [P, \partial_\mu P] = 0. \quad (3.36)$$

Now, let us define our submodel.

Definition 3.7.

$$[P, \square P] = 0, \quad (3.37)$$

$$[P, \partial_\mu P] \otimes \partial^\mu P + \partial^\mu P \otimes [P, \partial_\mu P] = 0. \quad (3.38)$$

We call these simultaneous equations the Grassmann submodel.

Remark 3.8. *In fact, (3.37) and (3.38) are equivalent to (3.35).*

Next, we express our submodel with $Z = (z_{kl})$ in (3.20).

Proposition 3.9. *The equations above are equivalent to*

$$\partial^\mu \partial_\mu Z = 0 \quad \text{and} \quad \partial^\mu Z \otimes \partial_\mu Z = 0 \quad (3.39)$$

or in each component

$$\partial^\mu \partial_\mu z_{kl} = 0 \quad \text{and} \quad \partial^\mu z_{kl} \partial_\mu z_{k'l'} = 0 \quad (3.40)$$

for any $1 \leq k, k' \leq N - j, 1 \leq l, l' \leq j$.

proof: By using (3.12) and (3.13), (3.38) is equivalent to the following equation.

$$\begin{aligned} & \begin{pmatrix} \partial_\mu Z^\dagger & \partial_\mu Z \end{pmatrix} \otimes \begin{pmatrix} \partial^\mu Z^\dagger & \partial^\mu Z \end{pmatrix} + \begin{pmatrix} \partial^\mu Z^\dagger & \partial^\mu Z \end{pmatrix} \otimes \begin{pmatrix} \partial_\mu Z^\dagger & \partial_\mu Z \end{pmatrix} \\ &= 2 \left(\begin{array}{c|c} & \partial_\mu Z^\dagger \otimes \partial^\mu Z^\dagger \\ \hline & 0 \\ \hline -\partial_\mu Z \otimes \partial^\mu Z & \end{array} \right) = 0, \end{aligned}$$

namely

$$\partial_\mu Z \otimes \partial^\mu Z = 0 \quad \text{and its Hermitian conjugate.} \quad (3.41)$$

From (3.23) and (3.41), we obtain the proposition. \square

In the case of the \mathbf{CP}^1 -submodel, we have

$$\partial^\mu \partial_\mu u = 0 \quad \text{and} \quad \partial^\mu u \partial_\mu u = 0 \quad (3.42)$$

with u in (3.26),(3.27). This is a generalization of that of [AFG1] because that is restricted to three dimensions.

Chapter 4

Conserved Currents of the Grassmann Submodel

4.1 Tensor Noether Currents

It is usually not easy to construct conserved currents except for Noether ones in the nonlinear Grassmann sigma models in any dimension, but in our submodels we can easily construct an infinite number of conserved currents. This is a feature typical of our submodels.

The equations of our submodel are

$$[P, \square P] = 0,$$

$$[P, \partial_\mu P] \otimes \partial^\mu P + \partial^\mu P \otimes [P, \partial_\mu P] = 0$$

in the global form (3.37),(3.38). Then we have

$$[\otimes^k P, \square(\otimes^k P)] = -[\square(\otimes^k P), \otimes^k P] = 0, \quad (4.1)$$

where

$$\otimes^k P \equiv \underbrace{P \otimes \cdots \otimes P}_k. \quad (4.2)$$

We show (4.1) in the case of $k = 3$ for simplicity. In general k , we can prove it in a similar way. If we put

$$[P, \partial_\mu P] = (a_{ij}), \quad \partial^\mu P = (b_{kl}), \quad P = (p_{mn}),$$

then (3.38) is

$$a_{ij}b_{kl} + b_{ij}a_{kl} = 0 \quad \text{for any } i, j, k, l \quad (4.3)$$

By using (3.37) and (3.38), we have only to prove

$$[P, \partial_\mu P] \otimes P \otimes \partial^\mu P + \partial^\mu P \otimes P \otimes [P, \partial_\mu P] = 0 \quad (4.4)$$

From (4.3), the $((ij), (mn), (kl))$ -component of (4.4) is

$$a_{ij}p_{mn}b_{kl} + b_{ij}p_{mn}a_{kl} = p_{mn}(a_{ij}b_{kl} + b_{ij}a_{kl}) = 0. \quad \square$$

By (4.1), we obtain the following theorem.

Theorem 4.1. *For $k = 1, 2, \dots$,*

$$\begin{aligned} & [\partial_\mu (\overset{k}{\otimes} P), \overset{k}{\otimes} P] \\ &= \sum_{i=0}^{k-1} \underbrace{P \otimes \dots \otimes P}_i \otimes [\partial_\mu P, P] \otimes \underbrace{P \otimes \dots \otimes P}_{k-1-i} \end{aligned} \quad (4.5)$$

are conserved currents of the Grassmann submodel.

Espcially, in the case of $k = 1$,

$$[\partial_\mu P, P] \quad (4.6)$$

is the original Noether current. We call (4.5) *the tensor Noether currents of degree k .*

Now, we write the matrix components of (4.5) in the case of the \mathbf{CP}^N -submodel. Let $\mathbf{u} = (u_1 \cdots u_N)^t$ be a local coordinate of \mathbf{CP}^N . Then, similar to the case of the Noether currents, we get the following proposition.

Proposition 4.2. *We use multi-index notations as follows:*

$$\mathbf{u}^{\mathbf{P}} = \mathbf{u}^{(p_1, \dots, p_N)} = u_1^{p_1} \dots u_N^{p_N}, \quad \bar{\mathbf{u}}^{\mathbf{Q}} = \bar{\mathbf{u}}^{(q_1, \dots, q_N)} = \bar{u}_1^{q_1} \dots \bar{u}_N^{q_N},$$

$$|\mathbf{P}| = p_1 + \dots + p_N, \quad |\mathbf{Q}| = q_1 + \dots + q_N.$$

Then the matrix components of (4.5) is

$$J_{(\mathbf{P}, \mathbf{Q}); \mu}^k = \frac{k(\partial_\mu \mathbf{u}^\dagger \mathbf{u} - \mathbf{u}^\dagger \partial_\mu \mathbf{u}) \mathbf{u}^{\mathbf{P}} \bar{\mathbf{u}}^{\mathbf{Q}} + (1 + \mathbf{u}^\dagger \mathbf{u})(\partial_\mu \mathbf{u}^{\mathbf{P}} \bar{\mathbf{u}}^{\mathbf{Q}} - \mathbf{u}^{\mathbf{P}} \partial_\mu \bar{\mathbf{u}}^{\mathbf{Q}})}{(1 + \mathbf{u}^\dagger \mathbf{u})^{k+1}} \quad (4.7)$$

$$0 \leq |\mathbf{P}| \leq k, 0 \leq |\mathbf{Q}| \leq k.$$

These are conserved currents of the \mathbf{CP}^N -submodel.

Proof: For

$$P = \frac{1}{1 + \mathbf{u}^\dagger \mathbf{u}} \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{u}^\dagger \end{pmatrix},$$

we put

$$\otimes^k \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} \equiv \begin{pmatrix} 1 \\ \mathbf{U}_k \end{pmatrix},$$

where

$$\mathbf{U}_k \equiv \underbrace{(\mathbf{u}, \dots, \mathbf{u})}_{kC_1} \underbrace{(\mathbf{u} \otimes \mathbf{u}, \dots, \mathbf{u} \otimes \mathbf{u})}_{kC_2}, \dots, \otimes^k \mathbf{u}^t.$$

Then, we have

$$\otimes^k P = \frac{1}{1 + \mathbf{U}_k^\dagger \mathbf{U}_k} \begin{pmatrix} 1 \\ \mathbf{U}_k \end{pmatrix} \begin{pmatrix} 1 & \mathbf{U}_k^\dagger \end{pmatrix}, \quad (4.8)$$

here we use relations

$$(\otimes^j \mathbf{u}^\dagger)(\otimes^j \mathbf{u}) = (\mathbf{u}^\dagger \mathbf{u})^j \quad (4.9)$$

and

$$1 + \mathbf{U}_k^\dagger \mathbf{U}_k = 1 + {}_kC_1 \mathbf{u}^\dagger \mathbf{u} + {}_kC_2 (\mathbf{u}^\dagger \mathbf{u})^2 + \dots + (\mathbf{u}^\dagger \mathbf{u})^k = (1 + \mathbf{u}^\dagger \mathbf{u})^k. \quad (4.10)$$

Therefore, we have only to substitute \mathbf{U}_k for \mathbf{u} in

$$\begin{aligned}
& [\partial_\mu P, P] \\
&= \frac{1}{(1 + \mathbf{u}^\dagger \mathbf{u})^2} \begin{pmatrix} J_\mu & J_\mu \mathbf{u}^\dagger - (1 + \mathbf{u}^\dagger \mathbf{u}) \partial_\mu \mathbf{u}^\dagger \\ J_\mu \mathbf{u} + (1 + \mathbf{u}^\dagger \mathbf{u}) \partial_\mu \mathbf{u} & J_\mu \mathbf{u} \mathbf{u}^\dagger + (1 + \mathbf{u}^\dagger \mathbf{u}) (\partial_\mu \mathbf{u} \mathbf{u}^\dagger - \mathbf{u} \partial_\mu \mathbf{u}^\dagger) \end{pmatrix},
\end{aligned} \tag{4.11}$$

where

$$J_\mu \equiv \partial_\mu \mathbf{u}^\dagger \mathbf{u} - \mathbf{u}^\dagger \partial_\mu \mathbf{u}.$$

First we note that

$$\begin{aligned}
\partial_\mu \mathbf{U}_k^\dagger \mathbf{U}_k &= {}_k C_1 \partial_\mu \mathbf{u}^\dagger \mathbf{u} + {}_k C_2 \partial_\mu (\mathbf{u}^\dagger \otimes \mathbf{u}^\dagger) (\mathbf{u} \otimes \mathbf{u}) + \cdots + \partial_\mu (\otimes^k \mathbf{u}^\dagger) (\otimes^k \mathbf{u}) \\
&= \partial_\mu \mathbf{u}^\dagger \mathbf{u} ({}_k C_1 + 2 {}_k C_2 \mathbf{u}^\dagger \mathbf{u} + \cdots + k (\mathbf{u}^\dagger \mathbf{u})^{k-1}) \\
&= k (1 + \mathbf{u}^\dagger \mathbf{u})^{k-1} \partial_\mu \mathbf{u}^\dagger \mathbf{u}
\end{aligned}$$

and

$$\mathbf{U}_k^\dagger \partial_\mu \mathbf{U}_k = k (1 + \mathbf{u}^\dagger \mathbf{u})^{k-1} \mathbf{u}^\dagger \partial_\mu \mathbf{u}.$$

Moreover, since the matrix component of $\partial_\mu \mathbf{U}_k \mathbf{U}_k^\dagger - \mathbf{U}_k \partial_\mu \mathbf{U}_k^\dagger$ are of the form

$$\partial_\mu \mathbf{u}^P \bar{\mathbf{u}}^Q - \mathbf{u}^P \partial_\mu \bar{\mathbf{u}}^Q, \tag{4.12}$$

we obtain the conserved currents (4.7). \square

Remark 4.3. *Another fomulation of our submodels is given by [FL]. This is a generalization of [AFG1] to the G/H -model in any dimension. They also construct a counterpart of our tensor Noether currents.*

4.2 Pull-Backed 1-Form

In this section, we investigate a generic form of conserved currents for our submodels by using a local coordinate of Grassmann manifolds. We write

$$Z = (z_1, \dots, z_{N'})^t, \quad N' = j(N - j) \quad (4.13)$$

as a vector in $\mathbf{C}^{N'}$ for simplicity. Now, we again write down our equations of the submodel:

$$\partial^\mu \partial_\mu Z = 0 \quad \text{and} \quad \partial^\mu Z \otimes \partial_\mu Z = 0. \quad (4.14)$$

$$\text{for } Z : M^{1+n} \longrightarrow \mathbf{C}^{N'}$$

or in each component

$$\partial^\mu \partial_\mu z_i = 0 \quad \text{and} \quad \partial^\mu z_i \partial_\mu z_j = 0 \quad (4.15)$$

for $1 \leq i, j \leq N'$.

We consider a 1-form on $\mathbf{C}^{N'}$;

$$\sum_{j=1}^{N'} \{f_j(Z, \bar{Z}) dz_j + g_j(Z, \bar{Z}) d\bar{z}_j\} \quad (4.16)$$

and pull back this on M^{1+n} by Z ;

$$Z : M^{1+n} \longrightarrow \mathbf{C}^{N'} \quad Z = (z_j).$$

Then, we look for conditions which make the pull-backed form a conserved current;

$$\sum_{j=1}^{N'} \partial^\mu \{f_j(Z, \bar{Z}) \partial_\mu z_j + g_j(Z, \bar{Z}) \partial_\mu \bar{z}_j\} = 0. \quad (4.17)$$

Making use of (4.15), we have

Proposition 4.4. *if the equations*

$$\frac{\partial f_j}{\partial \bar{z}_k} + \frac{\partial g_k}{\partial z_j} = 0 \quad (4.18)$$

for $1 \leq j, k \leq N'$ hold, then

$$\sum_{j=1}^{N'} \{f_j(Z, \bar{Z}) \partial_\mu z_j + g_j(Z, \bar{Z}) \partial_\mu \bar{z}_j\}. \quad (4.19)$$

is a conserved current of the Grassmann submodel.

In particular, if we set

$$f_j(Z, \bar{Z}) = \frac{\partial f}{\partial z_j}, \quad g_j(Z, \bar{Z}) = -\frac{\partial f}{\partial \bar{z}_j} \quad (4.20)$$

for any function f in C^2 -class, then we can easily check (4.18). Namely, we obtain conserved currents

$$\sum_{j=1}^{N'} \left(\frac{\partial f}{\partial z_j} \partial_\mu z_j - \frac{\partial f}{\partial \bar{z}_j} \partial_\mu \bar{z}_j \right) \quad (4.21)$$

parametrized by all C^2 -class functions f .

Proposition 4.4 is especially noteworthy in the case of the CP^1 -submodel.

In this case, (4.18) becomes

$$\frac{\partial f_1}{\partial \bar{u}} + \frac{\partial g_1}{\partial u} = 0.$$

Then the 1-form

$$f_1 du - g_1 d\bar{u}$$

is closed and there exists an $f = f(u, \bar{u})$ such that

$$f_1 = \frac{\partial f}{\partial u}, \quad g_1 = -\frac{\partial f}{\partial \bar{u}}$$

by the Poincaré lemma. Therefore we obtain the important corollary.

Corollary 4.5. *Let $J_{2,\mu}$ be a homogeneous differential polynomial of degree 1 with coefficient $C^2(\mathbf{C}, \mathbf{C})$. Then, $J_{2,\mu}$ is a conserved current of the \mathbf{CP}^1 -submodel if and only if there exist an $f = f(u, \bar{u})$ in $C^2(\mathbf{C}, \mathbf{C})$ such that*

$$J_{2,\mu} = \frac{\partial f}{\partial u} \partial_\mu u - \frac{\partial f}{\partial \bar{u}} \partial_\mu \bar{u}. \quad (4.22)$$

Remark 4.6. *We can express conserved currents in proposition 1.4 as (4.22) for suitable $f(u, \bar{u})$. For example, in the case of $m = j$,*

$$J_\mu^{(j,j)} = \left(\partial_\mu u \frac{\partial}{\partial u} - \partial_\mu \bar{u} \frac{\partial}{\partial \bar{u}} \right) \left(\frac{u^j}{(1 + |u|^2)^j} \right).$$

In particular, the Noether currents are

$$J_\mu^{Noet} = \frac{1}{(1 + |u|^2)^2} (\partial_\mu u \bar{u} - u \partial_\mu \bar{u}) = \left(\partial_\mu u \frac{\partial}{\partial u} - \partial_\mu \bar{u} \frac{\partial}{\partial \bar{u}} \right) \left(\frac{|u|^2}{1 + |u|^2} \right),$$

$$j_\mu = \frac{1}{(1 + |u|^2)^2} (\partial_\mu u + u^2 \partial_\mu \bar{u}) = \left(\partial_\mu u \frac{\partial}{\partial u} - \partial_\mu \bar{u} \frac{\partial}{\partial \bar{u}} \right) \left(\frac{u}{1 + |u|^2} \right),$$

and the complex conjugate \bar{j}_μ ,

or, by (4.11),

$$[\partial_\mu P, P] = \left(\partial_\mu u \frac{\partial}{\partial u} - \partial_\mu \bar{u} \frac{\partial}{\partial \bar{u}} \right) P, \quad (4.23)$$

where

$$P = \frac{1}{1 + |u|^2} \begin{pmatrix} 1 \\ u \end{pmatrix} \begin{pmatrix} 1 & \bar{u} \end{pmatrix}.$$

Moreover, the tensor Noether currents of degree k are also

$$\begin{aligned} J_{(p,q);\mu}^k &= \frac{k(\partial_\mu \bar{u} u - \bar{u} \partial_\mu u) u^p \bar{u}^q + (1 + |u|^2)(\partial_\mu u^p \bar{u}^q - u^p \partial_\mu \bar{u}^q)}{(1 + |u|^2)^{k+1}} \\ &= \left(\partial_\mu u \frac{\partial}{\partial u} - \partial_\mu \bar{u} \frac{\partial}{\partial \bar{u}} \right) \left(\frac{u^p \bar{u}^q}{(1 + |u|^2)^k} \right) \\ &\quad \text{for } 0 \leq p \leq k, 0 \leq q \leq k, \end{aligned}$$

namely,

$$[\partial_\mu (\binom{k}{\otimes} P), \binom{k}{\otimes} P] = \left(\partial_\mu u \frac{\partial}{\partial u} - \partial_\mu \bar{u} \frac{\partial}{\partial \bar{u}} \right) (\binom{k}{\otimes} P). \quad (4.24)$$

Chapter 5

Symmetries of the Grassmann Submodel

Our submodel has many good properties. If we construct solutions of our submodels, then we obtain a wide class of solutions of the nonlinear Grassmann models in any dimension, because, by definition, solutions of submodels are also those of the original models.

The equations of our submodels are

$$\partial^\mu \partial_\mu Z = 0 \quad \text{and} \quad \partial^\mu Z \otimes \partial_\mu Z = 0 \quad (3.39)$$

or in each component

$$\partial^\mu \partial_\mu z_{kl} = 0 \quad \text{and} \quad \partial^\mu z_{kl} \partial_\mu z_{k'l'} = 0 \quad (3.40)$$

for any $1 \leq k, k' \leq N - j, 1 \leq l, l' \leq j$.

Theorem 5.1. *The Grassmann submodel has a following symmetry; namely, if $z_{\alpha\beta}$ ($1 \leq \alpha \leq N - j, 1 \leq \beta \leq j$) is a solution of (3.40), then*

$$f_{kl}(z_{\alpha\beta}) = f_{kl}(z_{11}, \dots, z_{N-j,j}) \quad (5.1)$$

is also a solution of (3.40) for any holomorphic function f_{kl} on $\mathbf{C}^{j(N-j)}$ ($1 \leq k \leq N - j, 1 \leq l \leq j$).

Proof: Suppose that $z_{\alpha\beta}$ ($1 \leq \alpha \leq N - j, 1 \leq \beta \leq j$) satisfy (3.40). Then

$$\partial^\mu \partial_\mu f_{kl}(z_{\alpha\beta}) = \sum_{\alpha,\beta,\gamma,\delta} \left(\frac{\partial f_{kl}}{\partial z_{\alpha\beta}} \partial^\mu \partial_\mu z_{\alpha\beta} + \frac{\partial^2 f_{kl}}{\partial z_{\alpha\beta} \partial z_{\gamma\delta}} \partial_\mu z_{\alpha\beta} \partial^\mu z_{\gamma\delta} \right) = 0 \quad (5.2)$$

and

$$\partial^\mu f_{kl}(z_{\alpha\beta}) \partial_\mu f_{k'l'}(z_{\alpha'\beta'}) = \sum_{\alpha,\beta,\alpha',\beta'} \frac{\partial f_{kl}}{\partial z_{\alpha\beta}} \frac{\partial f_{k'l'}}{\partial z_{\alpha'\beta'}} \partial^\mu z_{\alpha\beta} \partial_\mu z_{\alpha'\beta'} = 0. \quad \square \quad (5.3)$$

For any k, l , $z_{kl} = \alpha_0 x_0 + \sum_{i=1}^n \alpha_i x_i$ with $\alpha^\mu \alpha_\mu \equiv \alpha_0^2 - \sum_{i=1}^n \alpha_i^2 = 0$ are clearly solutions of (3.40). Therefore, we obtain the following corollary.

Corollary 5.2. *Let f_{kl} ($1 \leq k \leq N - j, 1 \leq l \leq j$) be any holomorphic function. Then*

$$f_{kl}(\alpha_0 x_0 + \sum_{i=1}^n \alpha_i x_i) \quad (5.4)$$

under

$$\alpha^\mu \alpha_\mu = 0 \quad (5.5)$$

are solutions of our submodels.

Remark 5.3. *Symmetries discussed in this chapter become clearer in Part III.*

Part III

A Generalization of the Grassmann Submodel to Higher-Order Equations

Chapter 6

The Bell Polynomials

We shall generalize the equations of motion of the Grassmann submodel and the conserved currents to higher-order equations. In this chapter, we prepare a mathematical tool which plays an important role in our following theory.

6.1 The Bell Polynomials of One Variable

First, we describe the Bell polynomials of one variable in detail [Ri], [Ri1], [Ri2].

Definition 6.1. *Let $g(x)$ be a smooth function and z a complex parameter. Put $g_r \equiv \partial_x^r g(x)$. We define the Bell polynomials of degree n :*

$$F_n[zg] = F_n(zg_1, \dots, zg_n) \equiv e^{-zg(x)} \partial_x^n e^{zg(x)}. \quad (6.1)$$

The generating function of the Bell polynomials is formally written

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{F_n[zg]}{n!} t^n &= e^{-zg(x)} \sum_{n=0}^{\infty} \frac{(t\partial_x)^n}{n!} e^{zg(x)} \\ &= e^{-zg(x)} e^{zg(x+t)} \\ &= e^{z\{g(x+t)-g(x)\}} \end{aligned} \quad (6.2)$$

$$= \exp \left\{ z \sum_{j=1}^{\infty} \frac{g_j}{j!} t^j \right\}. \quad (6.3)$$

By (6.3), we can write $F_n[zg]$ explicitly as follows:

$$F_n(zg_1, \dots, zg_n) = \sum_{\substack{k_1+2k_2+\dots+nk_n=n \\ k_1 \geq 0, \dots, k_n \geq 0}} \frac{n!}{k_1! \dots k_n!} \left(\frac{zg_1}{1!}\right)^{k_1} \left(\frac{zg_2}{2!}\right)^{k_2} \dots \left(\frac{zg_n}{n!}\right)^{k_n}. \quad (6.4)$$

For example,

$$F_0 = 1, \quad F_1 = zg_1, \quad F_2 = zg_2 + z^2 g_1^2.$$

Remark 6.2. *These polynomials are used in the differential calculations of composite functions.*

$$\begin{aligned} & \partial_x^n(f(g(x))) \\ &= F_n[zg]|_{z=\frac{\partial}{\partial g}}(f) \end{aligned} \quad (6.5)$$

$$= \sum_{\substack{k_1+2k_2+\dots+nk_n=n \\ k_1 \geq 0, \dots, k_n \geq 0}} \frac{n!}{k_1! \dots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \dots \left(\frac{g_n}{n!}\right)^{k_n} \frac{\partial^{k_1+\dots+k_n} f}{\partial g^{k_1+\dots+k_n}} \quad (6.6)$$

(di Bruno's formula).

For example,

$$\partial_x(f(g(x))) = g_1 \frac{\partial f}{\partial g}, \quad \partial_x^2(f(g(x))) = g_2 \frac{\partial f}{\partial g} + g_1^2 \frac{\partial^2 f}{\partial g^2}. \quad (6.7)$$

Remark 6.3. *We put $z = 1$, $g(x) = e^x$. Then*

$$\exp(e^t - 1) = \exp(e^{x+t} - e^x)|_{x=0} = \sum_{n=0}^{\infty} \frac{F_n(1, 1, \dots, 1)}{n!} t^n \equiv \sum_{n=0}^{\infty} \frac{B(n)}{n!} t^n. \quad (6.8)$$

These numbers $B(n)$ are called the Bell numbers.

Remark 6.4. Let $S_n(x)$ be the elementary Schur polynomials which are defined by the generating function

$$\sum_{n=0}^{\infty} S_n(x) t^n = \exp \left\{ \sum_{j=0}^{\infty} x_j t^j \right\}. \quad (6.9)$$

Then we have

$$F_n[zg] = n! S_n \left(\frac{zg_1}{1!}, \frac{zg_2}{2!}, \dots \right). \quad (6.10)$$

Remark 6.5. The Bell polynomial is a generalization of the Hermite polynomial H_n , namely

$$H_n(x) = (-1)^n e^{x^2} \partial_x^n e^{-x^2} = (-1)^n F_n[-x^2]. \quad (6.11)$$

Lemma 6.6. We have a recursion formula for the Bell polynomials.

$$F_{n+1}[zg] = \left\{ \sum_{r=1}^n g_{r+1} \frac{\partial}{\partial g_r} + zg_1 \right\} F_n[zg]. \quad (6.12)$$

$$\begin{aligned} \text{Proof: } F_{n+1}[zg] &= e^{-zg(x)} \partial_x^{n+1} e^{zg(x)} \\ &= e^{-zg(x)} \partial_x (e^{zg(x)} e^{-zg(x)} \partial_x^n e^{zg(x)}) \\ &= (\partial_x + zg_1) F_n[zg] \\ &= \left\{ \sum_{r=1}^n \partial_x g_r \frac{\partial}{\partial g_r} + zg_1 \right\} F_n[zg]. \quad \square \end{aligned}$$

Next, we define the Bell matrix $B_{nj}[g] = B_{nj}(g_1, \dots, g_{n-j+1})$ (which is also called the Bell polynomials) by the following equation:

$$F_n(zg_1, \dots, zg_n) = \sum_{j \geq 0} z^j B_{nj}(g_1, \dots, g_{n-j+1}). \quad (6.13)$$

Note that

$$B_{n0} = \delta_{n0}, \quad B_{nj} = 0 \quad (n < j). \quad (6.14)$$

Remark 6.7. *We can express the Bell matrix as follows [Al-Fre]:*

$$B_{nj}(g_1, \dots, g_{n-j+1}) = \frac{1}{j!} \left\{ \frac{d^n}{dt^n} \left(\sum_{i=1}^{\infty} \frac{g_i}{i!} t^i \right)^j \right\}_{t=0}. \quad (6.15)$$

Then, by using lemma 6.6, we have a recursion formula for the Bell matrix.

Lemma 6.8. *For $n \geq 1$,*

$$B_{nj} = \sum_{r=1}^{n-j} g_{r+1} \frac{\partial}{\partial g_r} B_{n-1,j} + g_1 B_{n-1,j-1} \quad (j = 1, \dots, n-1), \quad (6.16)$$

$$B_{nn} = g_1 B_{n-1,n-1}. \quad (6.17)$$

In particular, we have

$$B_{n1}[g] = g_n, \quad B_{n,n-1}[g] = \frac{n(n-1)}{2} (g_1)^{n-2} g_2, \quad B_{nn}[g] = (g_1)^n. \quad (6.18)$$

An important formula for symmetries of higher-order \mathbf{CP}^1 -submodels is as follows:

Lemma 6.9. *The Bell matrix is anti-homomorphic with respect to the composition of functions, namely*

$$B_{pj}[f(g(x))] = \sum_{n=j}^p B_{pn}[g] B_{nj}[f]. \quad (6.19)$$

Proof:

$$\begin{aligned}
e^{z\{f(g(x+t))-f(g(x))\}} &= e^{z\{f(g(x)+(g(x+t)-g(x)))-f(g(x))\}} \\
&= e^{z\{f(g+w)-f(g)\}} \quad w \equiv g(x+t) - g(x) \\
&= \sum_{n=0}^{\infty} F_n[zf] \frac{w^n}{n!} \\
&= \sum_{n=0}^{\infty} F_n[zf] \frac{(g(x+t) - g(x))^n}{n!} \\
&= \sum_{n=0}^{\infty} F_n[zf] \sum_{p \geq n} B_{pn}[g] \frac{t^p}{p!} \\
&= \sum_{p=0}^{\infty} \sum_{p \geq n} B_{pn}[g] F_n[zf] \frac{t^p}{p!}.
\end{aligned}$$

Comparing both sides, we have

$$F_p[zf(g(x))] = \sum_{p \geq n} B_{pn}[g] F_n[zf],$$

namely

$$\sum_{p \geq j} B_{pj}[f(g(x))] z^j = \sum_{p \geq n} B_{pn}[g] \sum_{n \geq j} B_{nj}[f] z^j.$$

Therefore we obtain

$$B_{pj}[f(g(x))] = \sum_{p \geq n \geq j} B_{pn}[g] B_{nj}[f]. \quad \square$$

6.2 The Generalized Bell Polynomials

We call the Bell polynomials of multi-variables *the generalized Bell polynomials*[F].

Let $\mathbf{u}(\mathbf{y}) = (u_1(y_1, \dots, y_s), \dots, u_N(y_1, \dots, y_s))$ be smooth functions and $\mathbf{z} = (z_1, \dots, z_N)$ complex parameters. We use multi index notations

$$\mathbf{z}^{\mathbf{k}} = z_1^{k_1} \cdots z_N^{k_N}, \quad |\mathbf{k}| = k_1 + \cdots + k_N$$

and

$$\partial_{\mathbf{y}}^{\mathbf{p}} = \partial_{y_1}^{p_1} \cdots \partial_{y_s}^{p_s}, \quad |\mathbf{p}| = p_1 + \cdots + p_s.$$

Definition 6.10. *We define the generalized Bell polynomials:*

$$\begin{aligned} F_{\mathbf{p}}[\mathbf{z} \cdot \mathbf{u}] &= F_{p_1, \dots, p_s} \left[\sum_{i=1}^N z_i u_i(y_1, \dots, y_s) \right] \\ &\equiv e^{-\mathbf{z} \cdot \mathbf{u}(\mathbf{y})} \partial_{\mathbf{y}}^{\mathbf{p}} e^{\mathbf{z} \cdot \mathbf{u}(\mathbf{y})} \end{aligned} \quad (6.20)$$

$$\equiv \sum_{0 \leq |\mathbf{k}| \leq |\mathbf{p}|} \mathbf{z}^{\mathbf{k}} B_{\mathbf{p}, \mathbf{k}}[\mathbf{u}] \quad (6.21)$$

and we put $B_{\mathbf{p}, \mathbf{k}}[\mathbf{u}] = 0$ unless $0 \leq |\mathbf{k}| \leq |\mathbf{p}|$.

For example,

$$\begin{aligned} F_{100\dots 0} &= \sum_{i=1}^N z_i \partial_{y_1} u_i, \\ F_{200\dots 0} &= \sum_{i=1}^N z_i \partial_{y_1}^2 u_i + \left(\sum_{i=1}^N z_i \partial_{y_1} u_i \right)^2, \\ F_{110\dots 0} &= \sum_{i=1}^N z_i \partial_{y_1} \partial_{y_2} u_i + \left(\sum_{i=1}^N z_i \partial_{y_1} u_i \right) \left(\sum_{j=1}^N z_j \partial_{y_2} u_j \right). \end{aligned}$$

We often write $F_p = F_{p0\dots 0}$ and $(u_i)_r \equiv \partial_{y_1}^r u_i$ for simplicity.

Similar to lemma 6.6,

Lemma 6.11. *we have a recursion formula for the generalized Bell polynomials.*

$$F_{p+1}[\mathbf{z} \cdot \mathbf{u}] = \sum_{i=1}^N \left\{ \sum_{r=1}^p (u_i)_{r+1} \frac{\partial}{\partial (u_i)_r} + z_i (u_i)_1 \right\} F_p[\mathbf{z} \cdot \mathbf{u}]. \quad (6.22)$$

The generating function of the generalized Bell polynomials is

$$\begin{aligned} \sum_{\mathbf{p}} F_{\mathbf{p}}[\mathbf{z} \cdot \mathbf{u}] \frac{\mathbf{t}^{\mathbf{p}}}{\mathbf{p}!} &= \exp(\mathbf{z} \cdot \{\mathbf{u}(\mathbf{y} + \mathbf{t}) - \mathbf{u}(\mathbf{y})\}) \\ &= \prod_{i=1}^N \exp(z_i \{u_i(\mathbf{y} + \mathbf{t}) - u_i(\mathbf{y})\}). \end{aligned} \quad (6.23)$$

Therefore, similar to the proof of lemma 6.9,

Lemma 6.12. *we have*

$$B_{\mathbf{p}, \mathbf{j}}[\mathbf{f}(\mathbf{u}(\mathbf{y}))] = \sum_{|\mathbf{j}| \leq |\mathbf{m}| \leq |\mathbf{p}|} B_{\mathbf{p}, \mathbf{m}}[\mathbf{u}] B_{\mathbf{m}, \mathbf{j}}[\mathbf{f}], \quad (6.24)$$

where

$$\mathbf{f}(\mathbf{u}) = (f_1(u_1, \dots, u_N), \dots, f_N(u_1, \dots, u_N))$$

and

$$\mathbf{p} = (p_1, \dots, p_s), \quad \mathbf{m} = (m_1, \dots, m_N), \quad \mathbf{j} = (j_1, \dots, j_N).$$

Chapter 7

Higher-Order CP^1 -submodels

7.1 Definitions of Higher-Order CP^1 -submodels

In this section, we generalize the equation of motion of the CP^1 -submodel to higher-order equations. Hereafter, we use an notation of Minkowski summation as follows:

$$\sum_{\mu}' A_{\mu} \equiv A_0 - \sum_{j=1}^n A_j. \quad (7.1)$$

Remark 7.1. *The equations of the CP^1 -submodel*

$$\square_2 u \equiv \sum_{\mu}' \partial_{\mu}^2 u = 0 \quad \text{and} \quad \partial^{\mu} u \partial_{\mu} u = \sum_{\mu}' (\partial_{\mu} u)^2 = 0$$

are equivalent to

$$\square_2 u = 0 \quad \text{and} \quad \square_2(u^2) = 0. \quad (7.2)$$

Remark 7.2. *A solution u is called “a functionally invariant solution” if an arbitrary differentiable function $F(u)$ is also a solution of the same equation.*

See, for example, [KSG]. We find that u is a functionally invariant solution of the wave equation $\square_2 u = 0$ if and only if

$$\square_2 u = 0 \quad \text{and} \quad \square_2(u^2) = 0.$$

This is nothing but the \mathbf{CP}^1 -submodel.

We define higher-order nonlinear equations as follows:

Definition 7.3.

$$\square_p(u^k) \equiv \left(\frac{\partial^p}{\partial x_0^p} - \sum_{j=1}^n \frac{\partial^p}{\partial x_j^p} \right) (u^k) = 0 \quad \text{for } 1 \leq k \leq p. \quad (7.3)$$

We call this system of PDE the p -submodel.

Moreover, given $p = 2, 3, \dots$ and $i = 0, 1, \dots, [(p-1)/2]$, we also define

Definition 7.4.

$$\sum_{\mu} {}' \partial_{\mu}^{p-i}(u^k) \partial_{\mu}^i(\bar{u}^l) = 0 \quad (7.4)$$

for $k = 1, \dots, p-i$, $l = 0, \dots, i$.

We call this system of PDE the (p, i) -submodel.

Remark 7.5. In view of remark 7.1, these equations are equivalent to the \mathbf{CP}^1 -submodel in the case of $(p, i) = (2, 0)$.

Remark 7.6. We find that the (p, i) -submodel (7.4) is invariant under $u \rightarrow \frac{1}{u}$. Therefore we can express this equations by using P in \mathbf{CP}^1 . But it is difficult to find an “natural form” for any (p, i) . In the case of $(p, i) = (3, 0)$,

$$[P, \partial^{\mu} \partial_{\mu}^2 P - 3 \partial^{\mu} [P, [\partial_{\mu}^2 P]]] = 0, \quad (7.5)$$

$$[P \otimes P, \partial^{\mu} P \otimes [P, \partial_{\mu}^2 P] + [P, \partial_{\mu}^2 P] \otimes \partial^{\mu} P] = 0, \quad (7.6)$$

$$[P \otimes P \otimes P, \partial^{\mu} P \otimes \partial_{\mu} P \otimes \partial_{\mu} P] = 0. \quad (7.7)$$

7.2 Conserved Currents

In this section, we construct conserved currents for the equations (7.4).

let $g(x)$, $\bar{g}(x)$ be smooth functions and z , \bar{z} complex parameters. Put \mathcal{P}_B the vector space over \mathbf{C} spanned by the products of two Bell polynomials $F_n[zg]\bar{F}_m[\bar{z}\bar{g}]$. (We write $F_m[\bar{z}\bar{g}]$ as $\bar{F}_m[\bar{z}\bar{g}]$ for convinience.) We consider a linear map

$$\Phi : \mathcal{P}_B \rightarrow \mathbf{C}[\xi, \bar{\xi}], \quad (7.8)$$

$$\Phi(F_n[zg]\bar{F}_m[\bar{z}\bar{g}]) = \xi^n \bar{\xi}^m \quad (7.9)$$

We call $\xi^n \bar{\xi}^m$ “a symbol of $F_n[zg]\bar{F}_m[\bar{z}\bar{g}]$ ” and Φ “a symbol map”. By this map, \mathcal{P}_B is linear isomorphic to $\mathbf{C}[\xi, \bar{\xi}]$. Now, we define an operator

$$\partial \equiv \sum_{r=1}^{\infty} \left(g_{r+1} \frac{\partial}{\partial g_r} + \bar{g}_{r+1} \frac{\partial}{\partial \bar{g}_r} \right) + zg_1 + \bar{z}\bar{g}_1 \quad (7.10)$$

This operator is well-defined on \mathcal{P}_B and we have

$$\partial(F_n[zg]\bar{F}_m[\bar{z}\bar{g}]) = F_{n+1}[zg]\bar{F}_m[\bar{z}\bar{g}] + F_n[zg]\bar{F}_{m+1}[\bar{z}\bar{g}]. \quad (7.11)$$

In fact, because F_n is an n -variable polynomial and on account of (6.12),

$$\begin{aligned} & \partial(F_n[zg]\bar{F}_m[\bar{z}\bar{g}]) \\ &= \left\{ \sum_{r=1}^n g_{r+1} \frac{\partial}{\partial g_r} + \sum_{r=1}^m \bar{g}_{r+1} \frac{\partial}{\partial \bar{g}_r} + zg_1 + \bar{z}\bar{g}_1 \right\} F_n[zg]\bar{F}_m[\bar{z}\bar{g}] \quad (7.12) \\ &= \sum_{r=1}^n g_{r+1} \frac{\partial F_n[zg]}{\partial g_r} \bar{F}_m[\bar{z}\bar{g}] + zg_1 F_n[zg] \bar{F}_m[\bar{z}\bar{g}] \\ & \quad + F_n[zg] \sum_{r=1}^m \bar{g}_{r+1} \frac{\partial \bar{F}_m[\bar{z}\bar{g}]}{\partial \bar{g}_r} + \bar{z}\bar{g}_1 F_n[zg] \bar{F}_m[\bar{z}\bar{g}] \\ &= F_{n+1}[zg]\bar{F}_m[\bar{z}\bar{g}] + F_n[zg]\bar{F}_{m+1}[\bar{z}\bar{g}]. \end{aligned}$$

Because of (7.11), we have

$$\Phi \circ \partial \circ \Phi^{-1} = (\xi + \bar{\xi}), \quad (7.13)$$

where $(\xi + \bar{\xi})$ means the multiplication operator acting on $\mathbf{C}[\xi, \bar{\xi}]$.

By using the linear isomorphism Φ , we can identify

$$\begin{aligned} F_n \bar{F}_m & \text{ with } \xi^n \bar{\xi}^m, \quad \text{and} \\ \partial & \text{ with } (\xi + \bar{\xi}). \end{aligned} \quad (7.14)$$

Choose an $\mu \in \{0, \dots, n\}$ and put $x = x_\mu$, $g(x_\mu) = u(x_0, \dots, x_\mu, \dots, x_n)$. Then, we have $g_r = \partial_\mu^r u$. We set $F_{n,\mu}$ as

$$F_{n,\mu} \equiv : F_n(zg_1, \dots, zg_n)|_{z=\frac{\partial}{\partial u}} : \quad (7.15)$$

$$= : F_n(\partial_\mu u \frac{\partial}{\partial u}, \partial_\mu^2 u \frac{\partial}{\partial u}, \dots, \partial_\mu^n u \frac{\partial}{\partial u}) : \quad (7.16)$$

$$= \sum_{\substack{k_1+2k_2+\dots+nk_n=n \\ k_1 \geq 0, \dots, k_n \geq 0}} \frac{n!}{k_1! \dots k_n!} \left(\frac{\partial_\mu u}{1!} \right)^{k_1} \left(\frac{\partial_\mu^2 u}{2!} \right)^{k_2} \dots \left(\frac{\partial_\mu^n u}{n!} \right)^{k_n} \left(\frac{\partial}{\partial u} \right)^{k_1+k_2+\dots+k_n} \quad (7.17)$$

and $\bar{F}_{n,\mu}$ its complex conjugate of $F_{n,\mu}$, where $: \cdot :$ means the normal ordering.

Lemma 7.7.

$$\partial_\mu : F_{n,\mu} \bar{F}_{m,\mu} : f(u, \bar{u}) = : \partial(F_n[zg] \bar{F}_m[\bar{z} \bar{g}]) |_{z=\frac{\partial}{\partial u}} : f(u, \bar{u}). \quad (7.18)$$

where $f = f(u, \bar{u})$ is any function in C^{n+m+1} -class.

proof: If ∂_μ acts on a functional of the form

$$h(u, \partial_\mu u, \dots, \partial_\mu^n u; \bar{u}, \partial_\mu \bar{u}, \dots, \partial_\mu^m \bar{u}), \quad (7.19)$$

we can write

$$\partial_\mu = \sum_{r=1}^n \partial_\mu^{r+1} u \frac{\partial}{\partial(\partial_\mu^r u)} + \sum_{r=1}^m \partial_\mu^{r+1} \bar{u} \frac{\partial}{\partial(\partial_\mu^r \bar{u})} + \partial_\mu u \frac{\partial}{\partial u} + \partial_\mu \bar{u} \frac{\partial}{\partial \bar{u}}. \quad (7.20)$$

On the other hand, in view of (7.12) and since $g_r = \partial_\mu^r u$, $z = \frac{\partial}{\partial u}$, we have proved the lemma. \square

Lemma 7.8. *The (p, i) -submodel is equivalent to*

$$\sum_{\mu}' : F_{p-i, \mu} \bar{F}_{i, \mu} := 0. \quad (7.21)$$

proof: We note that

$$\begin{aligned} & \sum_{\mu}' \partial_{\mu}^{p-i}(u^k) \partial_{\mu}^i(\bar{u}^l) \\ &= \sum_{\mu}' \sum_{j_1=1}^{p-i} B_{p-i, j_1}(g_1, \dots, g_{p-i-j_1+1}) \left(\frac{\partial}{\partial u} \right)^{j_1} (u^k) \\ & \quad \times \sum_{j_2=0}^i B_{i, j_2}(\bar{g}_1, \dots, \bar{g}_{i-j_2+1}) \left(\frac{\partial}{\partial \bar{u}} \right)^{j_2} (\bar{u}^l) \\ &= \sum_{j_1=1}^{p-i} \sum_{j_2=0}^i j_1! j_2! \binom{k}{j_1} \binom{l}{j_2} \\ & \quad \times \sum_{\mu}' B_{p-i, j_1}(g_1, \dots, g_{p-i-j_1+1}) B_{i, j_2}(\bar{g}_1, \dots, \bar{g}_{i-j_2+1}) u^{k-j_1} \bar{u}^{l-j_2} \\ & \quad \text{for } k = 1, \dots, p-i, l = 0, \dots, i. \end{aligned}$$

Because of this, the (p, i) -submodel holds if and only if

$$\begin{aligned} & \sum_{\mu}' B_{p-i, j_1}(g_1, \dots, g_{p-i-j_1+1}) B_{i, j_2}(\bar{g}_1, \dots, \bar{g}_{i-j_2+1}) = 0 \quad (7.22) \\ & \quad \text{for } j_1 = 1, \dots, p-i, j_2 = 0, \dots, i, \end{aligned}$$

namely

$$\sum_{\mu}' : F_{p-i, \mu} \bar{F}_{i, \mu} := 0. \quad \square \quad (7.23)$$

In view of (7.14) and the lemmas above, we can search an infinite number of conserved currents as follows;

For fixed (p, i) , let W be the vector subspace of $\mathbf{C}[\xi, \bar{\xi}]$ spanned by $\{\xi^{p-i}\bar{\xi}^i, \xi^i\bar{\xi}^{p-i}\}$. Find the polynomials $p(\xi, \bar{\xi})$ such that

$$(\xi + \bar{\xi})p(\xi, \bar{\xi}) = 0 \quad \text{in } \mathbf{C}[\xi, \bar{\xi}]/W. \quad (7.24)$$

Then we can decide it uniquely (up to constant). That is

$$p(\xi, \bar{\xi}) = \sum_{k=0}^{p-1-2i} (-1)^k \xi^{p-1-i-k} \bar{\xi}^{i+k}. \quad (7.25)$$

Therefore, if we define the operator

$$V_{(p,i),\mu} \equiv \sum_{k=0}^{p-1-2i} (-1)^k : F_{p-1-i-k,\mu} \bar{F}_{i+k,\mu} :, \quad (7.26)$$

we obtain the next theorem.

Theorem 7.9. *For $p = 2, 3, \dots$ and $i = 0, 1, \dots, [(p-1)/2]$,*

$$V_{(p,i),\mu}(f) \quad (7.27)$$

are conserved currents for the (p, i) -submodel, where $f = f(u, \bar{u})$ is any function in C^p -class.

For example,

$$\begin{aligned} V_{(2,0),\mu}(f) &= F_{1,\mu}(f) - \bar{F}_{1,\mu}(f) \\ &= \partial_\mu u \frac{\partial f}{\partial u} - \partial_\mu \bar{u} \frac{\partial f}{\partial \bar{u}}, \end{aligned} \quad (7.28)$$

$$\begin{aligned} V_{(3,0),\mu}(f) &= F_{2,\mu}(f) - : F_{1,\mu} \bar{F}_{1,\mu} : (f) + \bar{F}_{2,\mu}(f) \\ &= \partial_\mu^2 u \frac{\partial f}{\partial u} + (\partial_\mu u)^2 \frac{\partial^2 f}{\partial u^2} - \partial_\mu u \partial_\mu \bar{u} \frac{\partial^2 f}{\partial u \partial \bar{u}} + \partial_\mu^2 \bar{u} \frac{\partial f}{\partial \bar{u}} + (\partial_\mu \bar{u})^2 \frac{\partial^2 f}{\partial \bar{u}^2}, \end{aligned} \quad (7.29)$$

$$\begin{aligned}
V_{(3,1),\mu}(f) &= : F_{1,\mu} \bar{F}_{1,\mu} : (f) \\
&= \partial_\mu u \partial_\mu \bar{u} \frac{\partial^2 f}{\partial u \partial \bar{u}}.
\end{aligned} \tag{7.30}$$

Hence, (7.27) is a generalization of (4.22).

We observe the role of the conserved current above. Let $J_{p,\mu}$ be a homogeneous differential polynomial of degree $(p-1)$ with coefficient $C^p(\mathbf{C}, \mathbf{C})$, namely

$$J_{p,\mu} = \sum_{n=0}^{p-1} \sum_{\substack{j \leq n \\ k \leq p-1-n}} f_{n,j,p-1-n,k} B_{nj;\mu}[u] B_{p-1-n,k;\mu}[\bar{u}], \tag{7.31}$$

where $f_{n,j,p-1-n,k} = f_{n,j,p-1-n,k}(u, \bar{u})$ in $C^p(\mathbf{C}, \mathbf{C})$ and

$$B_{nj;\mu}[u] \equiv B_{nj}[g]|_{g_r = \partial_\mu^r u}.$$

Remark 7.10. *In the case of $p = 2$,*

$$B_{11;\mu}[u] = \partial_\mu u, \quad B_{11;\mu}[\bar{u}] = \partial_\mu \bar{u}$$

and

$$J_{2,\mu} = f_{1100} \partial_\mu u + f_{0011} \partial_\mu \bar{u}.$$

See (4.19).

We put $V_{p,\mu} \equiv V_{(p,0),\mu}$ for simplicity. The next proposition is a generalization of corollary 4.5.

Theorem 7.11. *$J_{p,\mu}$ is a conserved current of the p -submodel if and only if there exist an $f = f(u, \bar{u})$ in $C^p(\mathbf{C}, \mathbf{C})$ such that*

$$J_{p,\mu} = V_{p,\mu}(f) + K_{p,\mu}, \tag{7.32}$$

where $K_{p,\mu}$ is a conserved current of the p -submodel which does not contain the “principal part” $\partial_\mu^n u \partial_\mu^{p-1-n} \bar{u}$ ($n = 0, \dots, p-1$). In particular, $K_{2,\mu} \equiv 0$.

proof: For convinience, we put $m = p - 1 - n$. By using the equations of the p -submodel, we find that $\partial^\mu J_{p,\mu} = 0$ is equivalent to the following equations;

$$\frac{\partial f_{n-1,n-1,m+1,m+1}}{\partial u} + \frac{\partial f_{n,n,m,m}}{\partial \bar{u}} = 0 \quad (\text{a})$$

$$\text{for } n = 1, \dots, p-1,$$

$$\frac{\partial f_{n-1,n-1,m+1,m}}{\partial u} - \frac{1}{m+1} \left(2f_{n,n,m,m} + (m-1) \frac{\partial f_{n,n,m,m-1}}{\partial \bar{u}} \right) = 0 \quad (\text{b})$$

$$\text{for } n = 1, \dots, p-2,$$

$$\frac{\partial f_{n+1,n,m-1,m-1}}{\partial \bar{u}} - \frac{1}{n+1} \left(2f_{n,n,m,m} + (n-1) \frac{\partial f_{n,n-1,m,m}}{\partial u} \right) = 0 \quad (\bar{\text{b}})$$

$$\text{for } m = 1, \dots, p-2,$$

$$\frac{\partial f_{n-1,n-1,m+1,k}}{\partial u} - f_{n,n,m,k} = 0 \quad (\text{c})$$

$$\text{for } n = 1, \dots, p-3; k = 1, \dots, m-1,$$

$$\frac{\partial f_{n+1,j,m-1,m-1}}{\partial \bar{u}} - f_{n,j,m,m} = 0 \quad (\bar{\text{c}})$$

$$\text{for } m = 1, \dots, p-3; j = 1, \dots, n-1,$$

$$\begin{aligned} & \frac{1}{n+1} \left(2f_{n,n,m,m-1} + (n-1) \frac{\partial f_{n,n-1,m,m-1}}{\partial u} \right) + \\ & \frac{1}{m} \left(2f_{n+1,n,m-1,m-1} + (m-2) \frac{\partial f_{n+1,n,m-1,m-2}}{\partial \bar{u}} \right) = 0 \end{aligned} \quad (\text{d})$$

$$\text{for } n = 1, \dots, p-3,$$

$$\frac{1}{n+1} \left(2f_{n,n,m,k} + (n-1) \frac{\partial f_{n,n-1,m,k}}{\partial u} \right) + f_{n+1,n,m-1,k} = 0 \quad (\text{e})$$

for $n = 1, \dots, p-4; k = 1, \dots, m-2,$

$$\frac{1}{m+1} \left(2f_{n,j,m,m} + (m-1) \frac{\partial f_{n,j,m,m-1}}{\partial \bar{u}} \right) + f_{n-1,j,m+1,m} = 0 \quad (\bar{e})$$

for $m = 1, \dots, p-4; j = 1, \dots, n-2,$

$$f_{n+1,j,m-1,k} + f_{n+2,j,m-2,k} = 0 \quad (f)$$

for $n = 1, \dots, p-5; j = 1, \dots, n; k = 1, \dots, m-3,$

$$\frac{\partial f_{n+1,j-1,m-1,k}}{\partial u} - f_{n+1,j,m-1,k} = 0 \quad (g)$$

for $n = 1, \dots, p-2; j = 2, \dots, n; k = 0, \dots, m-1,$

$$\frac{\partial f_{n-1,j,m+1,k-1}}{\partial \bar{u}} - f_{n-1,j,m+1,k} = 0 \quad (\bar{g})$$

for $n = 1, \dots, p-2; j = 0, \dots, n-1; k = 2, \dots, m.$

Solving these differential equations, we obtain the theorem 7.11. \square

7.3 Symmetries of Higher-Order CP^1 -submodels

7.3.1 A Generalization of Smirnov and Sobolev's Construction

In this section, we construct exact solutions of (7.4). We return to the CP^1 -submodel. It is defined by the equations

$$\partial^\mu \partial_\mu u = 0 \quad \text{and} \quad \partial^\mu u \partial_\mu u = 0 \quad (7.33)$$

for $u : M^{1+n} \longrightarrow \mathbf{C}.$

For the equations, a wide class of explicit solutions have been constructed by Smirnov and Sobolev (S-S in the following) [SS1], [SS2]. Let us make a short review according to [BY].

Let $a_\mu(u), b(u)$ be known functions and we consider an equation

$$0 = \delta \equiv \sum_{\mu=0}^n a_\mu(u) x_\mu - b(u) \quad (7.34)$$

with the constraint

$$\sum_{\mu} 'a_\mu(u)^2 = a_0(u)^2 - \sum_{j=1}^n a_j(u)^2 = 0. \quad (7.35)$$

Then, differentiating (7.34), we have easily

$$\partial_\mu u = -\frac{a_\mu}{\delta'}, \quad \text{where } ' = \frac{\partial}{\partial u}, \quad (7.36)$$

$$\partial_\mu^2 u = \frac{1}{\delta'^2} (2a_\mu a'_\mu - \frac{\delta''}{\delta'} a_\mu^2) = \frac{1}{\delta'^2} \left(\frac{\partial}{\partial u} - \frac{\delta''}{\delta'} \right) (a_\mu^2). \quad (7.37)$$

Remarking (7.35), we obtain (7.33). Next, we solve (7.34) making use of the inverse function theorem to be

$$u = u(x_0, x_1, \dots, x_n). \quad (7.38)$$

For example, if $a_\mu(u)$ is a constant;

$$a_0 x_0 + \sum_{j=1}^n a_j x_j - b(u) = 0 \quad (7.39)$$

with

$$a_0^2 - \sum_{j=1}^n a_j^2 = 0 \quad (7.40)$$

and b has its inverse (setting f), then we have the solutions (5.4)

$$u = f(a_0x_0 + \sum_{j=1}^n a_jx_j). \quad (7.41)$$

To generalize this method to our new higher-order equations, we prepare a theorem. For a smooth function $\delta = \delta(x_0, \dots, x_n, u)$ with $\delta' \equiv \frac{\partial \delta}{\partial u} \neq 0$, we consider an equation $\delta = 0$. Now, we define a linear operator

$$X \equiv -\frac{\partial}{\partial u} \circ \frac{1}{\delta'} = -\frac{1}{\delta'} \left(\frac{\partial}{\partial u} - \frac{\delta''}{\delta'} \right). \quad (7.42)$$

Then

Theorem 7.12. *we have*

$$\partial_\mu^p f(u) = -\frac{1}{\delta'} \sum_{j=1}^p X^{j-1} \left(\frac{\partial f}{\partial u} B_{pj}(\delta_1, \dots, \delta_{p-j+1}) \right) \quad (\mu = 0, \dots, n), \quad (7.43)$$

where $\delta_r = \partial_\mu^r|_{u:fix} \delta \quad (r = 1, \dots, p)$ and

$f : \mathbf{C} \longrightarrow \mathbf{C} : \text{holomorphic with respect to } u.$

.

Proof: We use the mathematical induction. First we have

$$\partial_\mu f(u) = -\frac{\delta_1}{\delta'} \frac{\partial f}{\partial u} = -\frac{1}{\delta'} \frac{\partial f}{\partial u} B_{11}(\delta_1). \quad (7.44)$$

Secondly, suppose that

$$\partial_\mu^{p-1} f(u) = -\frac{1}{\delta'} \sum_{j=1}^{p-1} X^{j-1} \left(\frac{\partial f}{\partial u} B_{p-1,j} \right), \quad (7.45)$$

then

$$\begin{aligned}
\partial_\mu^p f(u) &= -\frac{1}{\delta'} \sum_{j=1}^{p-1} \partial_\mu \left(X^{j-1} \left(\frac{\partial f}{\partial u} B_{p-1,j} \right) \right) + \frac{\partial_\mu \delta'}{\delta'^2} \sum_{j=1}^{p-1} X^{j-1} \left(\frac{\partial f}{\partial u} B_{p-1,j} \right) \\
&= -\frac{1}{\delta'} \sum_{j=1}^{p-1} \left\{ \partial_\mu \left(X^{j-1} \left(\frac{\partial f}{\partial u} B_{p-1,j} \right) \right) - \frac{1}{\delta'} \left(\frac{\partial \delta_1}{\partial u} - \frac{\delta_1}{\delta'} \delta'' \right) X^{j-1} \left(\frac{\partial f}{\partial u} B_{p-1,j} \right) \right\} \\
&= -\frac{1}{\delta'} \sum_{j=1}^{p-1} (\partial_\mu + X(\delta_1)) X^{j-1} \left(\frac{\partial f}{\partial u} B_{p-1,j} \right).
\end{aligned}$$

Noting that

$$[\partial_\mu + X(\delta_1), X] = 0 \quad (\text{by straightforward calculation}), \quad (7.46)$$

we have

$$\begin{aligned}
\partial_\mu^p f(u) &= -\frac{1}{\delta'} \sum_{j=1}^{p-1} X^{j-1} (\partial_\mu + X(\delta_1)) \left(\frac{\partial f}{\partial u} B_{p-1,j} \right) \\
&= -\frac{1}{\delta'} \sum_{j=1}^{p-1} X^{j-1} \left(\sum_{r=1}^{p-j} \delta_{r+1} \frac{\partial}{\partial \delta_r} - \frac{\delta_1}{\delta'} \frac{\partial}{\partial u} + X(\delta_1) \right) \left(\frac{\partial f}{\partial u} B_{p-1,j} \right) \\
&= -\frac{1}{\delta'} \sum_{j=1}^{p-1} X^{j-1} \left\{ \frac{\partial f}{\partial u} \sum_{r=1}^{p-j} \delta_{r+1} \frac{\partial}{\partial \delta_r} B_{p-1,j} + X \left(\frac{\partial f}{\partial u} \delta_1 B_{p-1,j} \right) \right\}.
\end{aligned}$$

By (6.16) and (6.17), we obtain

$$\partial_\mu^p f(u) = -\frac{1}{\delta'} \sum_{j=1}^p X^{j-1} \left(\frac{\partial f}{\partial u} B_{pj} \right). \quad \square \quad (7.47)$$

If we put $\delta \equiv \sum_{\mu=0}^n a_\mu(u) x_\mu - b(u)$, then, since $\delta_r = 0$ ($r \geq 2$),

$$B_{pj} = 0 \quad (1 \leq j \leq p-1), \quad B_{pp} = \delta_1^p = a_\mu^p. \quad (7.48)$$

Therefore

Corollary 7.13.

$$\partial_\mu^p f(u) = -\frac{1}{\delta'} X^{p-1} \left(\frac{\partial f}{\partial u} a_\mu^p \right). \quad (7.49)$$

Moreover, if we also put $f(u) = u$, then

Corollary 7.14.

$$\partial_\mu^p u = -\frac{1}{\delta'} X^{p-1} a_\mu^p. \quad (7.50)$$

This corollary is a generalization of (7.36), (7.37).

We remark that by theorem 7.12,

$$\begin{aligned} & \sum_{j=1}^p \sum_{\mu}' B_{pj}[u] \left(\frac{\partial}{\partial u} \right)^j f(u) \\ &= \sum_{\mu}' \partial_\mu^p f(u) \\ &= -\frac{1}{\delta'} \sum_{j=1}^p X^{j-1} \left(\frac{\partial f}{\partial u} \sum_{\mu}' B_{pj}[\delta] \right). \end{aligned} \quad (7.51)$$

Then we consider

$$\sum_{\mu}' B_{pj}[\delta] = 0 \quad (j = 1, \dots, p), \quad (7.52)$$

namely

$$\square_{p,x}(\delta^k(x, u)) \equiv \left(\frac{\partial^p}{\partial x_0^p} - \sum_{j=1}^n \frac{\partial^p}{\partial x_j^p} \right)_{u:\text{fix}} (\delta^k(x, u)) = 0 \quad \text{for } 1 \leq k \leq p. \quad (7.53)$$

These equations can be regarded as “*the p -submodel with parameter u* ”.

Therefore, if we consider a simple solution

$$\delta = \sum_{\mu=0}^n a_\mu(u) x_\mu - b(u) \quad \text{with} \quad \sum_{\mu}' a_\mu(u)^p = 0 \quad (7.54)$$

of (7.53) and put $\delta = 0$, we have

$$\sum_{\mu}' B_{pj}[u] = 0 \quad (j = 1, \dots, p), \quad (7.55)$$

namely, the p -submodel holds by (7.51).

Now, we also consider the (p, i) -submodel. Similarly, we put

$$0 = \delta \equiv \sum_{\mu=0}^n a_{\mu}(u)x_{\mu} - b(u) \quad (7.56)$$

with the constraint

$$\sum_{\mu}' a_{\mu}(u)^{p-i} \overline{a_{\mu}(u)^i} = 0. \quad (7.57)$$

By using theorem 7.12, then we have

$$\partial_{\mu}^{p-i}(u^k) = -\frac{1}{\delta'} X^{p-i-1} (k u^{k-1} a_{\mu}(u)^{p-i}), \quad (7.58)$$

$$\partial_{\mu}^i(\bar{u}^l) = -\frac{1}{\bar{\delta}'} \bar{X}^{i-1} (l \bar{u}^{l-1} \overline{a_{\mu}(u)^i}). \quad (7.59)$$

If we expand X^{p-i-1} and \bar{X}^{i-1} , and note that

$$\left(\frac{\partial}{\partial u}\right)^{n_1} a_{\mu}^{m_1} \left(\frac{\partial}{\partial \bar{u}}\right)^{n_2} \bar{a}_{\mu}^{m_2} = \left(\frac{\partial}{\partial u}\right)^{n_1} \left(\frac{\partial}{\partial \bar{u}}\right)^{n_2} a_{\mu}^{m_1} \bar{a}_{\mu}^{m_2}, \quad (7.60)$$

we obtain the next theorem.

Theorem 7.15. *We can construct exact solutions of (7.4) by using our extended S-S construction (7.56), (7.57).*

7.3.2 Symmetries and a Property of the Bell Matrix

In spite of higher-order equations, we find that all the solutions of the (p, i) -submodel are functionally invariant. This symmetry comes from the fact that the Bell matrix is anti-homomorphic with respect to the composition of functions.

Theorem 7.16. *If u is a solution of the (p, i) -submodel, then, for any holomorphic function f , $v = f(u)$ is also a solution of the (p, i) -submodel. In other words, all the solutions of the (p, i) -submodel are functionally invariant.*

proof: Suppose that u is a solution of the (p, i) -submodel. By lemma 7.8, the (p, i) -submodel is equivalent to

$$\sum_{\mu}' : F_{p-i, \mu} \bar{F}_{i, \mu} := 0, \quad (7.61)$$

namely

$$\sum_{\mu}' B_{p-i, j}[u] B_{ik}[\bar{u}] = 0 \quad \text{for } j = 1, \dots, p-i, \quad k = 0, \dots, i. \quad (7.62)$$

Therefore, by using of (6.19)

$$\begin{aligned} \sum_{\mu}' B_{p-i, j}[f(u)] B_{ik}[\overline{f(u)}] &= \sum_{\mu}' \sum_{n=j}^{p-i} \sum_{m=k}^i B_{p-i, n}[u] B_{nj}[f] B_{im}[\bar{u}] B_{mk}[\bar{f}] \\ &= 0. \end{aligned} \quad \square$$

For complex numbers a_{μ} ($\mu = 0, 1, \dots, n$) with $\sum_{\mu}' a_{\mu}^{p-i} \bar{a}_{\mu}^i = 0$,

$$u = a_0 x_0 + \sum_{i=1}^n a_i x_i \quad (7.63)$$

are clearly solutions of (7.4). Therefore, we obtain the following corollary.

Corollary 7.17. *Let f be any holomorphic function. Then*

$$f(a_0 x_0 + \sum_{i=1}^n a_i x_i) \quad (7.64)$$

under

$$\sum_{\mu}' a_{\mu}^{p-i} \bar{a}_{\mu}^i = 0 \quad (7.65)$$

are solutions of the (p, i) -submodel.

Chapter 8

A Higher-Order Grassmann submodel

8.1 A Definition of a Higher-Order Grassmann submodel

In this chapter, we generalize the equation of motion of the Grassmann submodel in a similar way to the \mathbf{CP}^1 -submodel by using the generalized Bell polynomial introduced in section 6.2.

We use the multi index notation.

Definition 8.1.

$$\square_p(\mathbf{u}^{\mathbf{k}}) = \square_p(u_1^{k_1} \cdots u_N^{k_N}) = 0 \quad \text{for } 1 \leq |\mathbf{k}| \leq p. \quad (8.1)$$

We call this system of PDE the \mathbf{CP}^N - p -submodel.

Remark 8.2. *In the case of $p = 2$, (8.1) is*

$$\square_2 u_i = 0 \quad \text{and} \quad \square_2(u_i u_j) = 0 \quad \text{for } i, j = 1, \dots, N.$$

These are equivalent to the \mathbf{CP}^N -submodel

$$\partial^\mu \partial_\mu u_i = 0 \quad \text{and} \quad \partial^\mu u_i \partial_\mu u_j = 0 \quad \text{for } i, j = 1, \dots, N.$$

Remark 8.3. By a local coordinate expression, the $G_{j,N}(\mathbf{C})$ -submodel and the $\mathbf{CP}^{j(N-j)}$ -submodel are equivalent. Therefore, (8.1) is also a generalization of the Grassmann submodel.

8.2 Conserved Currents

Lemma 8.4. The \mathbf{CP}^N - p -submodel has the Bell polynomial expression

$$\sum_{\mu} 'F_{p,\mu} = 0, \quad (8.2)$$

where

$$F_{p,\mu} \equiv \sum_{|\mathbf{j}| \leq p} B_{p;\mathbf{j};\mu}[\mathbf{u}] \left(\frac{\partial}{\partial \mathbf{u}} \right)^{\mathbf{j}} \quad (8.3)$$

and

$$B_{p;\mathbf{j};\mu}[\mathbf{u}] \equiv B_{(\underbrace{p,0,\dots,0}_{s-1},\mathbf{j})}[\mathbf{u}(\mathbf{y})] \Big|_{y_1=x_\mu, y_2,\dots,y_s:fix} \quad (8.4)$$

in (6.21).

Proof:

$$\begin{aligned} \square_p(\mathbf{u}^{\mathbf{k}}) &= \sum_{\mu} 'F_{p,\mu}(\mathbf{u}^{\mathbf{k}}) \\ &= \sum_{\mu} ' \sum_{|\mathbf{j}| \leq p} B_{p;\mathbf{j};\mu}[\mathbf{u}] \left(\frac{\partial}{\partial \mathbf{u}} \right)^{\mathbf{j}} (\mathbf{u}^{\mathbf{k}}) \\ &= \sum_{\mu} ' \sum_{|\mathbf{j}| \leq p} \mathbf{j}! \binom{\mathbf{k}}{\mathbf{j}} \sum_{\mu} ' B_{p;\mathbf{j};\mu}[\mathbf{u}] \mathbf{u}^{\mathbf{k}-\mathbf{j}} \quad \text{for } 1 \leq |\mathbf{k}| \leq p, \end{aligned}$$

where

$$\mathbf{j} = j_1! \cdots j_N! \quad \text{and} \quad \binom{\mathbf{k}}{\mathbf{j}} = \binom{k_1}{j_1} \cdots \binom{k_N}{j_N}.$$

Therefore, the \mathbf{CP}^N - p -submodel holds if and only if

$$\sum_{\mu} {}'B_{p,\mathbf{j};\mu}[\mathbf{u}] = 0 \quad \text{for} \quad 1 \leq |\mathbf{j}| \leq p, \quad (8.5)$$

namely

$$\sum_{\mu} {}'F_{p,\mu} = 0. \quad \square \quad (8.6)$$

By using the recursion formula of the generalized Bell polynomials (6.22), we can construct an infinite number of conserved currents for the \mathbf{CP}^N - p -submodel similar to the p -submodel.

Theorem 8.5.

$$V_{p,\mu}(f) \equiv \sum_{k=0}^{p-1} (-1)^k : F_{p-1-k,\mu} \bar{F}_{k,\mu} : (f) \quad (8.7)$$

are conserved currents for the \mathbf{CP}^N - p -submodel, where

$f = f(u_1, \dots, u_N, \bar{u}_1, \dots, \bar{u}_N)$ is any function in C^p -class.

For example,

$$\begin{aligned} V_{2,\mu}(f) &= F_{1,\mu}(f) - \bar{F}_{1,\mu}(f) \\ &= \sum_{i=1}^N \left(\partial_{\mu} u_i \frac{\partial f}{\partial u_i} - \partial_{\mu} \bar{u}_i \frac{\partial f}{\partial \bar{u}_i} \right), \end{aligned} \quad (8.8)$$

Hence, (8.7) is a generalization of (4.21).

8.3 Symmetries of the Higher-Order Grassmann submodel

As well as theorem 7.16, we find that all the solutions of the \mathbf{CP}^N - p -submodel are functionally invariant.

Theorem 8.6. *If u_1, \dots, u_N is a solution of the \mathbf{CP}^N - p -submodel, then $f_1(u_1, \dots, u_N), \dots, f_N(u_1, \dots, u_N)$ is also a solution of the \mathbf{CP}^N - p -submodel, where f_1, \dots, f_N are any holomorphic functions.*

Proof: Suppose that u_1, \dots, u_N is a solution of the \mathbf{CP}^N - p -submodel. By lemma 8.4, the \mathbf{CP}^N - p -submodel is equivalent to

$$\sum_{\mu} 'F_{p, \mu} = 0, \quad (8.9)$$

namely

$$\sum_{\mu} 'B_{(p, 0, \dots, 0), \mathbf{j}; \mu} [\mathbf{u}] = 0 \quad \text{for } 1 \leq |\mathbf{j}| \leq p. \quad (8.10)$$

Therefore, by using of (6.24)

$$\begin{aligned} \sum_{\mu} 'B_{(p, 0, \dots, 0), \mathbf{j}; \mu} [\mathbf{f}(\mathbf{u})] &= \sum_{\mu} ' \sum_{|\mathbf{j}| \leq |\mathbf{n}| \leq p} B_{(p, 0, \dots, 0), \mathbf{n}; \mu} [\mathbf{u}] B_{\mathbf{n}, \mathbf{j}; \mu} [\mathbf{f}] \\ &= 0 \quad \text{for } 1 \leq |\mathbf{j}| \leq p. \quad \square \end{aligned}$$

Bibliography

- [AFG1] O. Alvarez, L. A. Ferreira and J. S. Guillen: *A New Approach to Integrable Theories in Any Dimension*, Nucl. Phys. B529(1998)689-736, hep-th/9710147.
- [AFG2] O. Alvarez, L. A. Ferreira and J. S. Guillen: *Integrable theories in any dimension: a perspective*, hep-th/9903168.
- [AFZ1] H. Aratyn, L. A. Ferreira and A. H. Zimerman: *Toroidal solitons in 3+1 dimensional integrable theories*, Phys. Lett. B456(1999) 162-170, hep-th/9902141.
- [AFZ2] H. Aratyn, L. A. Ferreira and A. H. Zimerman: *Exact static soliton solutions of 3+1 dimensional integrable theory with nonzero Hopf numbers*, Phys. Rev. Lett. 83(1999) 1723-1726, hep-th/9905079.
- [Al-Fre] R. Aldrovandi and L. P. Freitas: *Continuous iteration of dynamical maps*, J. Math. Phys. 39(1998), 5324-5336.
- [B] Harold Blas: *Noether and topological currents equivalence and soliton/particle correspondence in affine $sl(2)^{(1)}$ Toda theory coupled to matter*, to appear in Nucl. Phys. B, hep-th/0011243.
- [Bell] E. T. Bell: *Exponential polynomials*, Ann. of Math. 35(1934), 258-277.

- [BY] A. F. Barannyk and I. I. Yuryk: *On a new method for constructing exact solutions of the nonlinear differential equations of mathematical physics*, J. Phys. A: Math. Gen. 31(1998), 4899-4907.
- [F] D. Fujiwara: *Generalized Bell polynomials*, (Japanese) Sugaku 42(1990), no. 1, 89-90.
- [FG] L. A. Ferreira and J. S. Guillen: *Infinite symmetries in the Skyrme model*, to appear in Phys. Lett. B, hep-th/0010168.
- [FHS1] K. Fujii, Y. Homma and T. Suzuki: *Nonlinear Grassmann Sigma Models in Any Dimension and An Infinite Number of Conserved Currents*, Phys. Lett. B438(1998)290-294, hep-th/9806084.
- [FHS2] K. Fujii, Y. Homma and T. Suzuki: *Submodels of Nonlinear Grassmann Sigma Models in Any Dimension and Conserved Currents, Exact Solutions*, Mod. Phys. Lett. A14(1999)919-928, hep-th/9809149.
- [FS1] K. Fujii and T. Suzuki: *Nonlinear Sigma Models in $(1+2)$ -Dimensions and An Infinite Number of Conserved Currents*, Lett. Math. Phys. 46(1998)49-59, hep-th/9802105.
- [FS2] K. Fujii and T. Suzuki: *Some Useful Formulas in Nonlinear Sigma Models in $(1+2)$ -Dimensions*, hep-th/9804004.
- [FL] L. A. Ferreira and E. E. Leite: *Integrable theories in any dimension and homogenous spaces*, Nucl. Phys. B547 (1999) 471-500, hep-th/9810067.
- [FR] L. A. Ferreira and A. V. Razumov: *Hopf solitons and area preserving diffeomorphisms of the sphere*, hep-th/0012176.

- [Fuj] K. Fujii: *Solutions of A_∞ Toda equations based on noncompact group $SU(1, 1)$ and infinite-dimensional Grassmann manifolds*, J. of Math. Phys. 36(1995), 1652-1665.
- [GMG] D. Giano, J. O. Madsen and J. S. Guillen: *Integrable Chiral Theories in 2+1 Dimensions*, Nucl. Phys. B537(1999)586-598, hep-th/9805094.
- [Jac] N. Jacobson: *Lie Algebras*, Interscience Publishers, John Wiley & Sons(1962).
- [KSG] N. S. Koshlyakov, M. M. Smirnov and E. B. Gliner: *Differential Equations of Mathematical Physics*, North-Holland Publishing Company, Amsterdam(1964)
- [Raj] R. Rajaraman: *Solitons and Instantons*, Elsevier(1982)
- [Ri] J. Riordan: *Derivatives of Composite Functions*, Bull. Am. Math. Soc. 52(1946), 664-667.
- [Ri1] J. Riordan: *An Introduction to Combinatorial Analysis*, John Wiley & Sons, New York(1958).
- [Ri2] J. Riordan: *Combinatorial Identities*, John Wiley & Sons, New York, London, Sydney(1968).
- [Suz] T. Suzuki: *A Generalization of the Submodel of Nonlinear \mathbf{CP}^1 Models*, Nucl. Phys. B578(2000)515-523, hep-th/0002003.
- [SS1] V. I. Smirnov and S. L. Sobolev: *New method for solving a plane problem of elastic oscillations*, Proc. of the Seismological Institute of the Academy of Sciences USSR, 20(1932), 37-42.

- [SS2] V. I. Smirnov and S. L. Sobolev: *On application of a new method to the study of elastic oscillations in a space with the axial symmetry*, Proc. of the Seismological Institute of the Academy of Sciences USSR, 29(1933), 43-51.
- [Zak] W. J. Zakrzewski: *Low Dimensional Sigma Models*, Adam Hilger (1989).

List of papers by Tatsuo Suzuki

- [FS1] K. Fujii and T. Suzuki: *Nonlinear Sigma Models in $(1+2)$ -Dimensions and An Infinite Number of Conserved Currents*, Lett. Math. Phys. 46(1998)49-59, hep-th/9802105.
- [FHS1] K. Fujii, Y. Homma and T. Suzuki: *Nonlinear Grassmann Sigma Models in Any Dimension and An Infinite Number of Conserved Currents*, Phys. Lett. B438(1998)290-294, hep-th/9806084.
- [FHS2] K. Fujii, Y. Homma and T. Suzuki: *Submodels of Nonlinear Grassmann Sigma Models in Any Dimension and Conserved Currents, Exact Solutions*, Mod. Phys. Lett. A14(1999)919-928, hep-th/9809149.
- [Suz] T. Suzuki: *A Generalization of the Submodel of Nonlinear \mathbf{CP}^1 Models*, Nucl. Phys. B578(2000)515-523, hep-th/0002003.